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APPLIED MECHANICS
(THERMOELASTICITY)

Certain Dynamic Problems of Thermoelasticity. III

by

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1. Introduction

In the present paper we are concerned with propagation of longitudinal thermoelastic waves in an infinite medium. Problems of this kind were already considered in [1]—[5] with reference to waves induced by the action of heat sources. Now our attention will be centered on the analysis of the effect of initial conditions on wave propagation.

To begin with, let us recall the differential equations describing the longitudinal waves [1]:

$$\nabla^2 \Phi - \frac{1}{c_1^2} \frac{\partial^2 \Phi}{\partial t^2} = m_0 \theta, \tag{1.1}$$

$$\nabla^2 \theta - \frac{1}{\varkappa} \frac{\partial \theta}{\partial t} - \eta_0 \nabla^2 \frac{\partial \Phi}{\partial t} = 0, \quad \nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}. \tag{1.2}$$

The first of these equations is the wave equation for the potential of thermoelastic displacement Φ, the second — the generalized equation of heat conduction. We are not concerned here with transversal waves: in an infinite medium they are independent of the longitudinal waves and are not related to the generation of heat.

The symbols used in Eqs. (1.1)—(1.2) denote: $\theta = T - T_0$ the difference between absolute temperature T and the temperature of natural state T_0, where the stresses as well as strains are equal to zero. $m_0 = \frac{\gamma}{\varrho_0 c_1^2}$, where $\gamma = (3\lambda + 2\mu)\,\alpha_t$. In the latter expression λ, μ are Lamé constants for the isothermic state and α_t is the coefficient of linear thermic dilatation; $c_1^2 = \frac{\lambda + 2\mu}{\varrho_0}$, ϱ_0 stands for density. $\varkappa = \frac{\lambda_0}{c_\varepsilon}$, where λ_0 denotes the coefficient of the heat conductivity, and c_ε is the specific heat at constant deformation referred to the volume unit. Finally $\eta = \frac{T_0 \gamma}{c_\varepsilon}$. The potential Φ and the temperature increment θ are functions of the point $\boldsymbol{x}$ and time t.

Eqs. (1.1) and (1.2) should be supplemented with the initial conditions

$$\Phi(x,0)=f(x), \frac{\partial\Phi(x,0)}{\partial t}=g_0(x), \quad \theta(x,0)=h(x). \tag{1.3}$$

After solving Eqs. (1.1) and (1.2) — the initial conditions (1.3) being taken into account — we may determine the displacements, deformations and stresses from the formulae

$$u_i=\Phi_{,i}, \quad \varepsilon_{ij}=\Phi_{,ij}, \quad \sigma_{ij}=2\mu(\Phi_{,ij}-\delta_{ij}\Phi_{,kk})+\varrho\delta_{ij}\ddot{\Phi}. \tag{1.4}$$

In the sequel it will be convenient to make use of the following new variables

$$\zeta_i=\frac{c_1}{\varkappa}x_i, \quad \tau=\frac{c_1^2}{\varkappa}t. \tag{1.5}$$

With these new variables, Eqs. (1.1) and (1.2) transform to

$$(\nabla^2-\partial_\tau^2)\Phi(\boldsymbol{\zeta},\tau)=m\theta(\boldsymbol{\zeta},\tau), \tag{1.6}$$

$$(\nabla^2-\partial_\tau)\theta(\boldsymbol{\zeta},\tau)-\eta\partial_\tau\nabla^2\Phi(\boldsymbol{\zeta},\tau)=0, \tag{1.7}$$

where

$$\nabla^2=\frac{\partial^2}{\partial\zeta_1^2}+\frac{\partial^2}{\partial\zeta_2^2}+\frac{\partial^2}{\partial\zeta_3^2}, \quad m=m_0\frac{\varkappa^2}{c_1^2}, \quad \eta=\eta_0\frac{c_1^2}{\varkappa},$$

and the initial conditions will be written in the form

$$\Phi(\boldsymbol{\zeta},0)=f(\boldsymbol{\zeta}), \quad \partial_\tau\Phi(\boldsymbol{\zeta},0)=g(\boldsymbol{\zeta}), \quad \theta(\boldsymbol{\zeta},0)=h(\boldsymbol{\zeta}), \quad g(\boldsymbol{\zeta})=\frac{\varkappa}{c_1^2}g_0(\boldsymbol{\zeta}). \tag{1.8}$$

Let us perform on Eqs. (1.6)—(1.8) the one-sided Laplace transformation as defined by the relation

$$\widetilde{\Phi}(\boldsymbol{\zeta},p)=\int_0^\infty \Phi(\boldsymbol{\zeta},\tau)e^{-p\tau}d\tau, \quad \text{etc.}$$

Thus, we obtain Eqs. (1.6) and (1.7) transformed to

$$(\nabla^2-p^2)\widetilde{\Phi}=m\tilde{\theta}-pf-g, \tag{1.9}$$

$$(\nabla^2-p)\tilde{\theta}-\eta p\nabla^2\widetilde{\Phi}=-h. \tag{1.10}$$

We eliminate from these equations first function $\widetilde{\Phi}$ and then function $\tilde{\theta}$. In this way we get the following equations

$$(\nabla^2-\lambda_1^2)(\nabla^2-\lambda_2^2)\widetilde{\Phi}=-mh-(\nabla^2-p)(pf+g), \tag{1.11}$$

$$(\nabla^2-\lambda_1^2)(\nabla^2-\lambda_2^2)\tilde{\theta}=-(\nabla^2-p^2)h-\eta p\nabla^2(pf+g). \tag{1.12}$$

Here, λ_1 and λ_2 are the roots of the equation:

$$\lambda^4-\lambda^2 p(p+1+\varepsilon)+p^3=0, \quad \varepsilon=\eta m,$$

hence,

$$\lambda_1^2, \lambda_2^2 = \frac{p}{2}\left[p + 1 + \varepsilon \pm \left(p^2 - 2p(1-\varepsilon) + (1+\varepsilon^2)\right)^{1/2}\right]. \tag{1.13}$$

From Eqs. (1.6) and (1.7) or (1.9) and (1.10) we can pass to the equations of the approximate theory, i.e. to what is called the theory of thermal stresses. We achieve this transition in a formal way, i.e. by putting $\eta = 0$. In this case, Eqs. (1.9) and (1.10) became mutually independent and the roots λ_1 and λ_2 take the values p and $p^{1/2}$, respectively.

2. Solutions of equations of thermoelasticity for the initial condition $\theta(\xi, 0) = h(\xi)$

Consider the auxiliary function G. Then the solution of the differential equation in an infinite thermoelastic space will read

$$[(\nabla^2 - \partial_\tau^2)(\nabla^2 - \partial_\tau) - \varepsilon\partial_\tau \nabla^2]\, G(\boldsymbol{\zeta}, \boldsymbol{\eta}; \tau) = -\,\delta(\boldsymbol{\zeta} - \boldsymbol{\eta})\,\delta(\tau). \tag{2.1}$$

The right-hand side of this equation represents a concentrated and instantaneous pulse applied at point $\boldsymbol{\eta}$. We assume the initial conditions of Eq. (2.1) to be homogeneous. After performing on Eq. (2.1) the Laplace transformation we get

$$(\nabla^2 - \lambda_1^2)(\nabla^2 - \lambda_2^2)\, \tilde{G}(\boldsymbol{\zeta}, \boldsymbol{\eta}, p) = -\,\delta(\boldsymbol{\zeta} - \boldsymbol{\eta}), \tag{2.2}$$

which is an equation of the same type as Eqs. (1.11) and (1.12).

The solution of Eq. (2.2) is known [1].

$$\tilde{G}(\boldsymbol{\zeta}, \boldsymbol{\eta}, p) = \frac{e^{-\lambda_1 \varrho} - e^{-\lambda_2 \varrho}}{4\pi\varrho\,(\lambda_1^2 - \lambda_2^2)}, \quad \varrho^2 = (\zeta_i - \eta_i)(\zeta_i - \eta_i), \quad i = 1, 2, 3. \tag{2.3}$$

This solution describes a spherical wave propagating from the point of disturbance towards infinity. The function $\tilde{G}(\boldsymbol{\zeta}, \boldsymbol{\eta}, p)$ will be of use in finding the solution of Eq. (1.2) for the case of prescribed initial conditions $0(\boldsymbol{\zeta}, 0) = h(\boldsymbol{\zeta})$. As a result we obtain the integral

$$\tilde{\Phi}(\boldsymbol{\eta}, p) = m \int_B h(\boldsymbol{\zeta})\, \tilde{G}(\boldsymbol{\zeta}, \boldsymbol{\eta}, p)\, dV(\boldsymbol{\zeta}). \tag{2.4}$$

The inverse Laplace transformation gives

$$\Phi(\boldsymbol{\eta}, \tau) = m \int_B h(\boldsymbol{\zeta})\, G(\boldsymbol{\zeta}, \boldsymbol{\eta}, \tau)\, dV(\boldsymbol{\zeta}). \tag{2.5}$$

Determination of the function G, i.e., the performance of the inverse Laplace transformation on the function (2.3) involves considerable mathematical difficulties. To avoid them, we choose to determine approximately the function G by the perturbation method. We take the quantity ε as the small parameter characterizing the coupling of the temperature field with that of deformation (cf., e.g., [2] and [6]). Assuming

$$\begin{aligned} \Phi &= \Phi_0 + \varepsilon\Phi_1 + \varepsilon^2 \Phi_2 + \ldots, \\ G &= G_0 + \varepsilon G_1 + \varepsilon^2 G_2 + \ldots. \end{aligned} \tag{2.6}$$

we obtain for successive approximations the following equation

$$\Phi_r(\boldsymbol{\eta}, \tau) = m \int_B h(\boldsymbol{\zeta})\, G_r(\boldsymbol{\zeta}, \boldsymbol{\eta}, \tau)\, dV(\boldsymbol{\zeta}), \qquad r = 0, 1, 2, \dots . \tag{2.7}$$

Only the first terms of the expanded form of Eq. (2.6) will be taken into account for the determination of the function Φ. Function G_r will be obtained by expanding the quantities $\lambda_1^2(\varepsilon, p)$, $\lambda_2^2(\varepsilon, p)$ and so on into the MacLaurin series following the ε powers and confining ourselves to the first power

$$\begin{gathered}
\lambda_1^2 \approx p^2 + \frac{p^2}{p-1}\varepsilon, \quad \lambda_2^2 \approx p - \frac{p}{p-1}\varepsilon, \\
\lambda_1 \approx p + \frac{p}{2(p-1)}\varepsilon, \quad \lambda_2 \approx p^{1/2}\left(1 - \frac{\varepsilon}{2(p-1)}\right). \\
\frac{1}{\lambda_1^2 - \lambda_2^2} \approx \frac{1}{p(p-1)}\left(1 - \varepsilon\frac{p+1}{(p-1)^2}\right).
\end{gathered} \tag{2.8}$$

Introducing λ_1 and λ_2 into the exponential terms $e^{-\lambda_1 \varrho}$, $e^{-\lambda_2 \varrho}$, we have

$$\begin{gathered}
e^{-\varrho\lambda_1} \approx \left(1 - \frac{\varepsilon\varrho}{2}\,\frac{p}{p-1}\right) e^{-\varrho p}, \\
e^{-\varrho\lambda_2} \approx \left(1 + \frac{\varepsilon\varrho}{2}\,\frac{p^{1/2}}{(p-1)}\right) e^{-\varrho p^{1/2}}.
\end{gathered} \tag{2.9}$$

Substituting (2.8) and (2.9) into (2.3), we get

$$\begin{gathered}
\widetilde{G}_0 = \frac{1}{4\pi\varrho}\,\frac{e^{-\varrho p} - e^{-\varrho p^{1/2}}}{p(p-1)}, \\
\widetilde{G}_1 = -\frac{1}{4\pi\varrho}\,\frac{1}{p(p-1)^2}\left[\left(\frac{p+1}{p-1} + \frac{\varrho p}{2}\right) e^{-\varrho p} - \left(\frac{p+1}{p-1} + \frac{\varrho p^{1/2}}{2}\right) e^{-\varrho p^{1/2}}\right].
\end{gathered} \tag{2.10}$$

The inverse Laplace transformation gives

$$\begin{aligned}
G_0(\varrho, \tau) &= \frac{1}{4\pi\varrho}\left\{(e^{\tau-\varrho} - 1)\, H(\tau - \varrho) - \left[U(\varrho, \tau) - \operatorname{erfc}\frac{\varrho}{2\sqrt{\tau}}\right]\right\}, \\
G_1(\varrho, \tau) &= -\frac{1}{4\pi\varrho}\left\{\left[\left[(\tau-\varrho)^2 + (\tau-\varrho)\left(\frac{\varrho}{2} - 1\right) + 1\right] e^{\tau-\varrho} - 1\right] \times \right. \\
&\quad \times H(\tau - \varrho) + \alpha(\varrho, \tau)\frac{\varrho}{2} - \left[\frac{1}{2}\left(\tau^2 + \frac{\varrho^2}{4}\right) U + \frac{1}{2}\left(\frac{\varrho}{4} - \tau\varrho\right) V - \right. \\
&\quad \left. - \frac{\varrho}{4}\left(\frac{\tau}{\pi}\right)^{1/2} \exp\left(-\frac{\varrho^2}{4\tau}\right)\right] - \left[\frac{\tau^2}{2} - \tau + \frac{\varrho^2}{8} + 1\right] U + \left(\frac{5}{8}\varrho - \frac{\tau\varrho}{2}\right) V + \\
&\quad \left. + \operatorname{erfc}\left(\frac{\varrho}{2\sqrt{\tau}}\right) - \frac{\varrho}{4}\left(\frac{\tau}{\pi}\right)^{1/2} \exp\left(-\frac{\varrho^2}{4\tau}\right)\right\}.
\end{aligned} \tag{2.12}$$

The following notations were introduced

$$U = \frac{e^{\tau}}{2}\left[e^{-\varrho}\operatorname{erfc}\left(\frac{\varrho}{2\sqrt{\tau}} - \sqrt{\tau}\right) + e^{\varrho}\operatorname{erfc}\left(\frac{\varrho}{2\sqrt{\tau}} + \sqrt{\tau}\right)\right],$$

$$(2.13) \qquad V = \frac{e^{\tau}}{2}\left[e^{-\varrho}\operatorname{erfc}\left(\frac{\varrho}{2\sqrt{\tau}} - \sqrt{\tau}\right) - e^{\varrho}\operatorname{erfc}\left(\frac{\varrho}{2\sqrt{\tau}} + \sqrt{\tau}\right)\right],$$

$$a(\varrho, \tau) = \int_0^{\tau}\left[\left(\tau_0 + \frac{1}{2}\right)V(\varrho, \tau_0) - \frac{\varrho}{2}U(\varrho, \tau_0) + \left(\frac{\tau_0}{\pi}\right)^{1/2}\exp\left(-\frac{\varrho^2}{4\tau_0}\right)\right]d\tau_0.$$

Function $G_0(\boldsymbol{\zeta}, \boldsymbol{\eta}, \tau)$ refers to the non-coupled problem, representing a spherical wave, its center being at point $\boldsymbol{\eta}$. The first term $e^{\varepsilon-\varrho}H(\tau-\varrho)$ is the expression for an elastic wave propagating from point η towards infinity with the velocity $c_1 = \left(\frac{\lambda+2\mu}{\varrho_0}\right)^{1/2}$. The second term, $U(\varrho, \tau) - \operatorname{erfc}\left(\frac{\varrho}{2\sqrt{\tau}}\right)$ is of diffusional character. Similar terms are characteristic also of the function $G_1(\boldsymbol{\zeta}, \boldsymbol{\eta}, \tau)$.

Introducing G_0 and G_1 into (2.4), we obtain the expression for the function Φ in the following form

$$(2.14) \qquad \Phi(\boldsymbol{\eta}, \tau) = m\int_B h(\boldsymbol{\zeta})\,[G_0(\boldsymbol{\zeta}, \boldsymbol{\eta}, \tau) + \varepsilon G_1(\boldsymbol{\zeta}, \boldsymbol{\eta}, \tau)]\,dV(\boldsymbol{\zeta}),$$

which is the approximate solution of Eq. (1.11), provided

$$\theta(\boldsymbol{\zeta}, 0) = h(\boldsymbol{\zeta}), \quad g(\boldsymbol{\zeta}) = 0, \quad f(\boldsymbol{\zeta}) = 0.$$

To find the temperature $\theta(\boldsymbol{\eta}, \tau)$ which will give the solution of Eq. (1.12) for the initial condition $\theta(\boldsymbol{\zeta}, 0) = h(\boldsymbol{\zeta})$ we have to consider the auxiliary function $F(\boldsymbol{\zeta}, \boldsymbol{\eta}, \tau)$ satisfying the equation

$$(2.15) \qquad [(\nabla^2 - \partial_\tau^2)(\nabla^2 - \partial_\tau) - \varepsilon\partial_\tau\nabla^2]\,F(\boldsymbol{\zeta}, \boldsymbol{\eta}, \tau) = -(\nabla^2 - \partial_\tau^2)\,\delta(\boldsymbol{\zeta} - \boldsymbol{\eta})\,\delta(\tau).$$

Assuming the initial conditions for Eq. (2.15) to be homogeneous, we obtain — after performing on this equation the Laplace transformation —

$$(2.16) \qquad \bar{F}(\boldsymbol{\zeta}, \boldsymbol{\eta}, p) = \frac{(\lambda_1^2 - p^2)\,e^{-\varrho\lambda_1} - (\lambda_2^2 - p^2)\,e^{-\varrho\lambda_2}}{4\pi\varrho\,(\lambda_1^2 - \lambda_2^2)},$$

$$\varrho^2 = (\zeta_i - \eta_i)(\zeta_i - \eta_i), \quad i = 1, 2, 3.$$

Function $\theta(\boldsymbol{\eta}, \tau)$ giving the solution of Eq. (1.12) may be expressed in the form of the integral

$$(2.17) \qquad \theta(\boldsymbol{\eta}, \tau) = \int_B h(\boldsymbol{\zeta})\,F(\boldsymbol{\zeta}, \boldsymbol{\eta}, \tau)\,dV(\boldsymbol{\zeta}).$$

Similarly as was done for the function G, we expand the term $F(\boldsymbol{\zeta}, \boldsymbol{\eta}, \tau)$ into a power series after ε so as to obtain

$$(2.18) \qquad \theta(\boldsymbol{\eta}, \tau) = \int_B h(\boldsymbol{\zeta})\,[F_0(\boldsymbol{\zeta}, \boldsymbol{\eta}, \tau) + \varepsilon F_1(\boldsymbol{\zeta}, \boldsymbol{\eta}, \tau) + \ldots]\,dV(\boldsymbol{\zeta}).$$

Making use of relations (2.8) and (2.9), we have

$$\tag{2.19} \begin{aligned} \widetilde{F}_0(\varrho, p) &= \frac{e^{-\varrho p^{1/2}}}{4\pi\varrho}, \\ \widetilde{F}_1(\varrho, p) &= \frac{p}{4\pi\varrho(p-1)^2}\left[e^{-\varrho p} + \left(\frac{\varrho}{2}p^{-1/2}(p-1) - 1\right)e^{-\varrho p^{1/2}}\right]. \end{aligned}$$

The inverse Laplace transformation performed on the above formulae yields:

$$\tag{2.20} F_0(\varrho, \tau) = \frac{1}{8(\pi\tau)^{3/2}} \exp\left(-\frac{\varrho^2}{4\tau}\right),$$

$$\tag{2.21} \begin{aligned} F_1(\varrho, \tau) = \frac{1}{4\pi\varrho}\Bigg\{(\tau - \varrho + 1)\, e^{\tau-\varrho} H(\tau - \varrho) - \Bigg[(\tau + 1)\, U(\varrho, \tau) - \\ - \frac{\varrho}{2} V(\varrho, \tau)\Bigg] + \frac{\varrho}{2}\left[V(\varrho, \tau) + \left(\frac{1}{\tau\pi}\right)^{1/2} \exp\left(-\frac{\varrho^2}{4\tau}\right)\right]\Bigg\}. \end{aligned}$$

Function F_0 is of diffusional character; it refers to the non-coupled problem, the theory of thermal stresses. The expression for the function F_1 contains both a wave and a diffusion term.

3. Solution of equations of thermoelasticity for the initial conditions $\Phi(\zeta, 0) = f(\zeta)$ and $\partial_\tau \Phi(\zeta, 0) = g(\zeta)$

We now turn our attention to the auxiliary function $K(\zeta, \eta, \tau)$ which is the solution of the wave equation

$$\tag{3.1} [(\nabla^2 - \partial_\tau^2)(\nabla^2 - \partial_\tau) - \varepsilon\partial_\tau \nabla^2]\, K(\zeta, \eta, \tau) = -(\nabla^2 - \partial_\tau)\, \delta(\zeta - \eta)\, \delta(\tau),$$

provided the initial conditions relating to Eq. (3.1) are assumed to be homogeneous. The solution of Eq. (3.1) after performing the Laplace transformation is of the following form

$$\tag{3.2} \widetilde{K}(\zeta, \eta, p) = \frac{(\lambda_1^2 - p)\, e^{-\varrho\lambda_1} - (\lambda_2^2 - p)\, e^{-\varrho\lambda_2}}{4\pi\varrho(\lambda_1^2 - \lambda_2^2)}.$$

We introduce additionally, besides the function K, the function $L(\zeta, \eta, \tau)$ as the particular integral of the equation

$$\tag{3.3} [(\nabla^2 - \partial_\tau^2)(\nabla^2 - \partial_\tau) - \varepsilon\partial_\tau \nabla^2]\, L(\zeta, \eta, \tau) = -\nabla^2 \delta(\zeta - \eta)\, \delta(\tau).$$

The initial conditions for the function $L(\zeta, \eta, \tau)$ are assumed to be homogeneous.

The Laplace transformation of the solution of Eq. (3.3) is given by the formula

$$\tag{3.4} \widetilde{L}(\zeta, \eta, p) = \frac{\lambda_1^2 e^{-\lambda_1\varrho} - \lambda_2^2 e^{-\lambda_2\varrho}}{4\pi\varrho(\lambda_1^2 - \lambda_2^2)}.$$

The functions K and L being known, we are now able to solve Eq. (1.11) and (1.12) for the prescribed initial conditions $\Phi(\zeta, 0) = f(\zeta)$, $\partial_\tau \Phi(\zeta, 0) = g(\zeta)$. The following functions are the solutions of Eqs. (1.11) and (1.12)

$$\tilde{\Phi}(\eta, p) = \int_B [pf(\zeta) + g(\zeta)]\, \tilde{K}(\zeta, \eta, p)\, dV(\zeta), \tag{3.5}$$

$$\tilde{\theta}(\eta, p) = \frac{\varepsilon p}{m} \int_B [pf(\zeta) + g(\zeta)]\, \tilde{L}(\zeta, \eta, p)\, dV(\zeta). \tag{3.6}$$

We shall expand the expressions for the functions $\tilde{K}$ and $\tilde{L}$

$$\begin{aligned} \tilde{K} &= \tilde{K}_0 + \varepsilon \tilde{K}_1 + \varepsilon^2 \tilde{K}_2 + \dots, \\ \tilde{L} &= \tilde{L}_0 + \varepsilon \tilde{L}_1 + \varepsilon^2 \tilde{L}_2 + \dots . \end{aligned} \tag{3.7}$$

Introducing (3.7) into (3.5) and (3.6) and retaining only the terms of the expansions of (3.5) which are linear with respect to ε, we obtain the following approximate formulae for the functions $\tilde{\Phi}$ and $\tilde{\theta}$.

$$\tilde{\Phi}(\eta, p) = \int_B [pf(\zeta) + g(\zeta)]\, [\tilde{K}_0(\zeta, \eta, p) + \varepsilon \tilde{K}_1(\zeta, \eta, p)]\, dV(\zeta), \tag{3.8}$$

$$\tilde{\theta}(\eta, p) = \frac{\varepsilon p}{m} \int_B [pf(\zeta) + g(\zeta)]\, \tilde{L}_0(\zeta, \eta, p)\, dV(\zeta). \tag{3.9}$$

Taking into account the relations (2.8) and (2.9) and expanding formulae (3.2) and (3.4) into series according to the powers of ε we have

$$\begin{aligned} &\tilde{K}_0(\varrho, p) = \frac{1}{4\pi\varrho} e^{-\varrho p}, \quad \tilde{L}_0(\varrho, p) = \frac{pe^{-\varrho p} - e^{-\varrho p^{1/2}}}{4\pi\varrho(p-1)}, \\ &\tilde{K}_1(\varrho, p) = -\frac{1}{4\pi\varrho}\left[\frac{1}{(p-1)^2}(pe^{-\varrho p} - e^{-\varrho p^{1/2}}) + \frac{\varrho}{2}\,\frac{-p}{p-1} e^{-\varrho p}\right], \dots . \end{aligned} \tag{3.10}$$

Let us now consider the first approximation of the functions $\tilde{\Phi}$ and $\tilde{\theta}$ obtained from the formulae (3.5) and (3.6).

$$\tilde{\Phi}_0(\eta, p) = \int_B [pf(\zeta) + g(\zeta)]\, e^{-\varrho p}\, dV(\zeta), \quad \tilde{\theta}_0(\eta, p) = 0. \tag{3.11}$$

We perform the inverse Laplace transformation on Eq. (3.11) and we get

$$\Phi_0(\eta, \tau) = \int_B [g(\zeta)\,\delta(\tau - \varrho) + f(\zeta)\,\delta'(\tau - \varrho)]\, dV(\zeta), \quad \theta_0(\eta, \tau) = 0. \tag{3.12}$$

We introduce the spherical coordinates $(\varrho, \varphi, \vartheta)$, where

$$0 \leqslant \varphi < 2\pi, \quad 0 \leqslant \vartheta < \pi, \quad \varrho^2 = (\zeta_i - \eta_i)(\zeta_i - \eta_i), \quad i = 1, 2, 3.$$

Taking point $\boldsymbol{\eta}$ as the centre of the sphere we have

$$\zeta_i = \eta_i + n_i \varrho,$$

where

$$n_1 = \sin\vartheta\cos\varphi, \quad n_2 = \sin\vartheta\sin\varphi, \quad n_3 = \cos\vartheta,$$

are the cosine directionals of the straight line connecting the centre of the sphere $\boldsymbol{\eta}$ with point $\boldsymbol{\zeta}$.

Since in the spherical coordinates there is $dV = \varrho^2\, d\varrho \sin\vartheta d\vartheta d\varphi$, we reduce the relation (3.12) to the form

$$\Phi_0(\boldsymbol{\eta}, \tau) = \frac{1}{4\pi}\int_B [g(\eta_i + n_i\varrho)\,\delta(\tau-\varrho) + f(\eta_i + n_i\varrho)\,\delta'(\tau-\varrho)]\times \\ \times \varrho d\varrho \sin\vartheta d\vartheta d\varphi = \tau M_{\eta,\tau}\{g\} + \frac{\partial}{\partial\tau}[M_{\eta,\tau}\{f\}]. \tag{3.13}$$

Here the quantity

$$M_{\eta,\tau}\{g\} = \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_0^{\pi} g(\eta_i + n_i\tau)\sin\vartheta d\vartheta \tag{3.14}$$

was introduced; it denotes the mean value of the function g on the surface of a sphere with centre at η and with radius τ. The solution (3.13) is known from the classical theory of elastokinetics [7].

The second approximation for the function Φ will be obtained from Eq. (3.8). It reads

$$\Phi_1 = \int_B [g(\boldsymbol{\zeta})\,K_1(\boldsymbol{\zeta}, \boldsymbol{\eta}, \tau) + f(\boldsymbol{\zeta})\,\partial_\tau K_1(\boldsymbol{\zeta}, \boldsymbol{\eta}, \tau)]\,dV(\boldsymbol{\zeta}). \tag{3.15}$$

Taking into account the relations (3.10), we get

$$K_1(\varrho, \tau) = -\frac{1}{4\pi\varrho}\left\{\left(\tau - \frac{\varrho}{2} + 1\right)e^{\tau-\varrho}H(\tau-\varrho) + \frac{\varrho}{2}\delta(\tau-\varrho) - \right. \\ \left. -\left[\tau U(\varrho,\tau) - \frac{\varrho}{2}V(\varrho,\tau)\right]\right\}, \tag{3.16}$$

$$\partial_\tau K_1(\varrho, \tau) = -\frac{1}{4\pi\varrho}\left\{\left(\tau - \frac{\varrho}{2} + 2\right)e^{\tau-\varrho}H(\tau-\varrho) + \left(1 + \frac{\varrho}{2}\right)\delta(\tau-\varrho) + \right. \\ \left. + \frac{\varrho}{2}\delta'(\tau-\varrho) - \partial_\tau\left[\tau U(\varrho,\tau) - \frac{\varrho}{2}V(\varrho,\tau)\right]\right\}. \tag{3.17}$$

In the expressions for the functions K_1 and $\partial_\tau K_1$ we have terms of diffusional type as well as terms characterizing a spherical elastic wave propagating from point $\boldsymbol{\eta}$ towards infinity.

Introducing (3.16) and (3.17) into (3.15) and making use of surface integrals (3.14), we arrive at the following formula

$$(3.18)\qquad \Phi_1(\boldsymbol{\eta},\tau) = -\frac{1}{4\pi}\int\limits_{\tau\geqslant\varrho}\frac{dV(\boldsymbol{\zeta})}{\varrho(\boldsymbol{\zeta},\boldsymbol{\eta})}\Big[g(\boldsymbol{\zeta})\Big(\tau-\frac{\varrho}{2}+1\Big)+ +f(\boldsymbol{\zeta})\Big(\tau-\frac{\varrho}{2}+2\Big)\Big]e^{\tau-\varrho}+\frac{1}{4\pi}\int\limits_{\tau\geqslant\varrho}\frac{dV(\boldsymbol{\zeta})}{\varrho(\boldsymbol{\zeta},\boldsymbol{\eta})}\Big\{g(\boldsymbol{\zeta})\Big[\tau U(\varrho,\tau)- -\frac{\varrho}{2}V(\varrho,\tau)\Big]+f(\boldsymbol{\zeta})\Big[(1+\tau)U(\varrho,\tau)-\frac{\varrho}{2}V(\varrho,\tau)\Big]\Big\}-\tau M_{\eta,\tau}\{f\}- -\frac{1}{2}\tau^2[M_{\eta,\tau}\{f+g\}]+\frac{1}{2}\partial_\tau[\tau^2 M_{\eta,\tau}\{f\}].$$

We now have to determine the second approximation for the temperature. After the inverse Laplace transformation on Eq. (3.9) we obtain

$$(3.19)\qquad \theta(\boldsymbol{\eta},\tau)=\varepsilon\theta_1(\boldsymbol{\eta},\tau)=\frac{\varepsilon}{m}\int\limits_B[g(\boldsymbol{\zeta})\,\partial_\tau L_0(\boldsymbol{\zeta},\boldsymbol{\eta},\tau)+ +f(\boldsymbol{\zeta})\,\partial_\tau^2 L_0(\boldsymbol{\zeta},\boldsymbol{\eta},\tau)]\,dV(\boldsymbol{\zeta})=\frac{\varepsilon}{4\pi m}\int\limits_B\frac{g(\boldsymbol{\zeta})}{\varrho(\boldsymbol{\zeta},\boldsymbol{\eta})}\Big\{\delta'(\tau-\varrho)+\delta(\tau-\varrho)+ +e^{\tau-\varrho}H(\tau-\varrho)-\Big[U(\varrho,\tau)-\frac{\varrho}{2\sqrt{\pi\tau^3}}e^{-\frac{\varrho^2}{4\tau}}\Big]\Big\}dV(\boldsymbol{\zeta})+ +\frac{\varepsilon}{4\pi m}\int\limits_B\frac{f(\boldsymbol{\zeta})}{\varrho(\boldsymbol{\zeta},\boldsymbol{\eta})}\Big\{\delta''(\tau-\varrho)+\delta'(\tau-\varrho)+\delta(\tau-\varrho)+e^{\tau-\varrho}H(\tau-\varrho)- -\partial_\tau\Big[U(\varrho,\tau)+\frac{\varrho}{2\sqrt{\pi\tau^3}}e^{-\frac{\varrho^2}{4\tau}}\Big]\Big\}dV(\boldsymbol{\zeta}).$$

This expression may be reduced to the form

$$(3.20)\qquad \theta(\boldsymbol{\eta},\tau)=\frac{\varepsilon}{4\pi m}\int\limits_{\tau\geqslant\varrho}\frac{dV(\boldsymbol{\zeta})}{\varrho(\boldsymbol{\zeta},\boldsymbol{\eta})}[g(\boldsymbol{\zeta})+f(\boldsymbol{\zeta})]\,e^{\tau-\varrho}- -\frac{\varepsilon}{4\pi m}\int\limits_V\frac{dV(\boldsymbol{\zeta})}{\varrho(\boldsymbol{\zeta},\boldsymbol{\eta})}\Big\{g(\boldsymbol{\zeta})\Big[U(\varrho,\tau)+\frac{\varrho}{2\sqrt{\pi\tau^3}}e^{-\frac{\varrho^2}{4\tau}}\Big]+ +f(\boldsymbol{\zeta})\,\partial_\tau\Big[U(\varrho,\tau)+\frac{\varrho}{2\sqrt{\pi\tau^3}}e^{-\frac{\varrho^2}{4\tau}}\Big]\Big\}+\tau M_{\eta,\tau}\{g+f\}- -\partial_\tau[\tau M_{\eta,\tau}\{g+f\}]-\partial_\tau^2[\tau M_{\eta,\tau}\{f\}].$$

Let us remark that the formula (3.20) contains diffusional terms as well as those characteristic for the propagation of spherical waves. The full text of the present work including uni- and two-dimensional problems will be published in Proceedings of Vibration Problems.

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В. НОВАЦКИЙ, **НЕКОТОРЫЕ ДИНАМИЧЕСКИЕ ПРОБЛЕМЫ ТЕРМОУПРУГОСТИ. III.**

Предметом работы являются проблемы расхождения продольных термоупругих волн в неограниченной среде. Заданные начальные условия рассматриваются в качестве импульсов, вызывающих пропагацию упомянутых волн. Используя соответствующие функции Грина, волновые уравнения и применяя метод возмущений получается приближенное решение для потенциала Φ и температуры θ в замкнутом виде,