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APPLIED MECHANICS

The Reciprocity Theorem in Magneto-thermo-elasticity. I.

by

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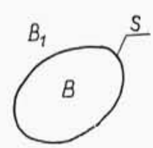
1. Introduction

The results hitherto obtained in the domain of magneto-thermo-elasticity are the establishment of the general equations for real and perfect conductors, [1], and the obtainment of a number of solutions concerning principally the propagation of discontinuity waves in one-dimensional problems for perfect and real conductors [2], [3] and also the propagation of plane waves [4], [5]. A number of more general solutions have been obtained in problems of magneto-elasticity.

In the present paper are derived general relations constituting a generalization of the familiar Betti's reciprocity theorem to the case of a magneto-thermo-elastic field. The paper is confined to the case of perfect elastic conductors postponing for the time being the consideration of analogous problems for real conductors and anelastic bodies. Sec. **2** of the present paper gives the fundamental equations and Sec. **3** — the derivation of the reciprocity theorem.

2. Fundamental equations

Let us consider the equations of magneto-thermo-elasticity in the case of a perfect elastic conductor. In this case these equations can be reduced to a set of equations with two dependent variables which are the displacement and the temperature. By solving this set of equations it is easy to calculate the perturbed values of the field $\vec{h}$ and $\vec{E}$ and the state of stresses and strains.



Let us consider a region B bounded by a surface S in the period $t > 0$ (Figure). The set of equations of the problem is composed of

a) the equation of motion:

$$\sigma_{ij,j} + T_{ij,j} + X_i = \varrho \ddot{u}_i, \tag{2.1}$$

b) the heat equation:

$$\left(\nabla^2 - \frac{1}{\varkappa}\partial_t\right)\theta - \eta\dot{e} = -\frac{Q}{\varkappa}, \tag{2.2}$$

c) the equations of electrodynamics for a slowly moving body:

$$\text{rot}\,\vec{h} = \frac{4\pi}{c}\vec{j}; \quad \text{rot}\,\vec{E} = -\frac{\mu_0}{c}\dot{\vec{h}}; \quad \vec{E} = -\frac{\mu_0}{c}(\dot{\vec{u}}\times\vec{H}); \quad \text{div}\,\vec{h} = 0. \tag{2.3}$$

hence

$$\vec{h} = \text{rot}\,(\vec{u}\times\vec{H}),$$

$$\vec{j} = \frac{c}{4\pi}\,\text{rot}\,\text{rot}\,(\vec{u}\times\vec{H}),$$

where:

$$T_{ij} = \frac{\mu_0}{4\pi}\left[h_i H_j + h_j H_i - \delta_{ij}(\vec{h}\cdot\vec{H})\right] \tag{2.4}$$

is Maxwell's tensor (the electric components of this tensor have fallen off due to linearization), X_i — body force components, ϱ — density, $\gamma = (3\lambda+2\mu)\,a_t$, where λ, μ are isothermal elastic constants, a_t — the coefficient of linear thermal dilatation. Next, $\varkappa = \lambda_0/\varrho c_\varepsilon$, where λ_0 is the coefficient of heat conduction and c_ε is the specific heat with constant strain; $\eta = \gamma T_0/\lambda_0$, where T_0 is the absolute temperature in the natural state, in which the strains and the stresses are equal to zero and, finally, $Q = W/\varrho c_\varepsilon$, where W is the quantity of heat produced per unit volume and time. The symbol $e = \varepsilon_{kk}$ denotes the dilatation, c — the velocity of light in vacuum and μ_0 — the magnetic permeability which is usually assumed, for perfect non-ferromagnetic conductors, to be equal to unity. The stresses are related to the strains and the temperature by the Duhamel—Neumann relations

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + (\lambda e - \gamma\theta)\,\delta_{ij}, \tag{2.5}$$

where

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}).$$

If the body bounded by the surface S is not in contact with a medium filling up B_1 outside B, these equations are completed with the equations of the field in vacuum in the region B_1 (Figure).

$$\text{rot}\,\vec{h}^* = \frac{1}{c}\dot{\vec{E}}^*; \quad \text{rot}\,\vec{E}^* = -\frac{1}{c}\dot{\vec{h}}^* \tag{2.6}$$

or

$$\left(\nabla^2 - \frac{1}{c^2}\partial_t^2\right)\vec{E}^*, \vec{h}^* = 0.$$

It will be assumed in what follows that the primary magnetic field $\vec{H}$ is directed along the x_3 axis, that is $\vec{H} = (0, 0, \text{H})$. This means, of course, no loss of generality.

With this assumption we find from (2.3):

$$E_1 = -\frac{\mu_0 H}{c}\dot{u}_2; \quad E_2 = \frac{\mu_0 H}{c}\dot{u}_1; \quad E_3 = 0; \tag{2.7}$$

(2.8) $\dot{h}_1 = \frac{c}{\mu_0} E_{2,3}; \quad \dot{h}_2 = -\frac{c}{\mu_0} E_{1,3}; \quad \dot{h}_3 = -\frac{c}{\mu_0}(E_{2.1} - E_{1,2});$

(2.9) $j_1 = \frac{c}{4\pi}(h_{3,2} - h_{2,3}); \quad j_2 = \frac{c}{4\pi}(h_{1,3} - h_{3,1}); \quad j_3 = \frac{b}{4\pi}(h_{2,1} - h_{1,2}).$

On calculating the components of Maxwell's tensor and inserting them into the equations of motion (2.1), we obtain the set of equations

$$\begin{aligned} &\mu\nabla^2 u_1 + (\lambda+\mu+\varrho a_0^2)\, e_{,1} + \varrho a_0^2 (u_{1,3} - u_{3,1})_{,3} + X_1 - \gamma\theta_{,1} = \varrho \ddot{u}_1, \\ &\mu\nabla^2 u_2 + (\lambda+\mu+\varrho a_0^2)\, e_{,2} + \varrho a_0^2 (u_{2,3} - u_{3,2})_{,3} + X_2 - \gamma\theta_{,2} = \varrho \ddot{u}_2, \qquad (2.10) \\ &\mu\nabla^2 u_3 + (\lambda+\mu)\, e_{,3} + X_3 - \gamma\theta_{,3} = \varrho \ddot{u}_3, \end{aligned}$$

where $a_0^2 = \mu_0^2 H^2/4\pi\varrho$, a_0 is the Alfvèn velocity.

The set of Eqs. (2.10) and (2.2) together with (2.3) constitute a complete set of equations of magneto-thermo-elasticity for a perfect conductor. Let us observe that the matrix of operators in the set of Eqs. (2.10) is symmetric. Hence Green's functions obtained from (2.2) and (2.10) will also be symmetric. Let us observe in addition that the coupling between the electromagnetic field and the strain and temperature field is effected through the factor a_0^2, depending on $H_3 = H$. For $H \to 0$ the set of Eqs. (2.10) becomes that of equations of thermoelasticity. Eqs. (2.10) show symmetry in relation to the x_3-axis. The structure of these equations bears some resemblance to that of a transversally isotropic elastic body. The resemblance is of a formal nature, however, because Maxwell's tensor is not expressed in terms of the strain tensor [6], [7], it thus being possible to obtain, by means, for instance, of a judicious choice of constants expressing the influence of the field $\vec{H}$, equations of the same structure as the equations of elasticity for a transversally isotropic body.

The above set of equations should be completed with the boundary conditions at the surface S, where displacements or loads $p_i = (\sigma_{ij} + T_{ij} - T_{ij}^*)\, n_j$ may be prescribed, T_{ij}^* denoting Maxwell's tensor of electromagnetic field intensity in vacuum and the continuity conditions of tangent electric fields and normal magnetic induction. In addition the temperature θ should be prescribed or the temperature gradient in the direction normal to S. If the action of the vacuum is neglected $T_{ij}^* = 0$ and the continuity conditions of the electromagnetic field fall off. It will be assumed in what follows that every cause producing motion starts its action at the moment $t = 0^+$. The initial conditions are assumed to be homogeneous.

3. The reciprocity theorem

Let us perform the Laplace transformation on all the equations of Sec. **2** assuming homogeneous initial conditions. We consider two sets of causes and effects. We start out from the identity (2.5):

(3.1) $(\bar{\sigma}_{ij} + \gamma\bar{\theta}\delta_{ij})\, \bar{\varepsilon}'_{ij} = (\bar{\sigma}'_{ij} + \gamma\bar{\theta}'\,\delta_{ij})\, \bar{\varepsilon}_{ij} = (2\mu\bar{\varepsilon}_{ij} + \lambda\bar{e}\delta_{ij})\, \bar{\varepsilon}'_{ij} = (2\mu\bar{\varepsilon}'_{ij} + \lambda\bar{e}'\,\delta_{ij})\, \bar{\varepsilon}_{ij},$

where

$$\bar{\sigma}_{ij} = L(\sigma_{ij}) = \int_0^\infty e^{-pt}\, \sigma_{ij}(x, t)\, dt, \qquad \text{etc.}$$

On integrating the expression (3.1) over the region B, we find

$$(3.2)\qquad \int_B (\bar{\sigma}_{ij}\bar{\varepsilon}'_{ij} - \bar{\sigma}'_{ij}\bar{\varepsilon}_{ij})\,dV = \int_B (\bar{\sigma}_{ij}\bar{u}'_{i,j} - \bar{\sigma}'_{ij}\bar{u}_{i,j})\,dV = \int_B [(\bar{\sigma}_{ij}\bar{u}'_i)_{,j} + \\ - \bar{\sigma}_{ij,j}\bar{u}'_i - (\bar{\sigma}'_{ij}\bar{u}_{i,j})_{,j} + \bar{\sigma}'_{ij,j}\bar{u}_i]\,dV = -\gamma\int_B (\bar{\theta}\bar{e}' - \bar{\theta}'\bar{e})\,dV.$$

Making use of the divergence theorem and the equation (2.1), and performing the Laplace transformation, we obtain

$$(3.3)\qquad \int_B (\bar{X}_i\bar{u}'_i - \bar{X}'_i\bar{u}_i)\,dV + \int_S (\bar{p}_i\bar{u}'_i - \bar{p}'_i\bar{u}_i)\,dS + \gamma\int_B (\bar{\theta}\bar{e}' - \bar{\theta}'\bar{e})\,dV = \\ = \int_B (\bar{T}_{ij}\bar{\varepsilon}'_{ij} - \bar{T}'_{ij}\bar{\varepsilon}_{ij})\,dV,$$

where $\bar{p}_i$ is expressed in a different manner depending on whether the action of the vacuum has or has not been taken account of. In order to simplify the considerations we neglect the action of the field in vacuum. However, in conclusion of the present section additional equations will be given in order to take into consideration this influence. Disregarding the influence of the field in vacuum, we have

$$\bar{p}_i = (\bar{\sigma}_{ij} + \bar{T}_{ij})\,n_j; \quad \bar{p}'_i = (\bar{\sigma}'_{ij} + \bar{T}'_{ij})\,n_j.$$

Making use of Eq. (2.2,) we find

$$(3.4)\qquad \int_B (\bar{\theta}'\nabla^2\bar{\theta} - \bar{\theta}\nabla^2\bar{\theta}')\,dV = \eta p\int_B (\bar{e}\bar{\theta}' - \bar{e}'\bar{\theta})\,dV - \frac{1}{\varkappa}\int_B (\bar{Q}\bar{\theta}' - \bar{Q}'\bar{\theta})\,dV.$$

Using Green's identity, we transform (3.4) to obtain

$$(3.5)\qquad \int_B (\bar{e}\bar{\theta}' - \bar{e}'\bar{\theta})\,dV = \frac{1}{\varkappa\eta p}\int_B (\bar{Q}\bar{\theta}' - \bar{Q}'\bar{\theta})\,dV + \frac{1}{\eta p}\int_S (\bar{\theta}'\bar{\theta}_{,n} - \bar{\theta}\bar{\theta}'_{,n})\,dS.$$

Finally, substituting (3.5) into (3.3), we obtain the reciprocity equation:

$$(3.6)\qquad \eta\varkappa p\left[\int_B (\bar{X}_i\bar{u}'_i - \bar{X}'_i\bar{u}_i)\,dV + \int_S (\bar{p}_i\bar{u}'_i - \bar{p}'_i\bar{u}_i)\,dS\right] = \gamma\int_B (\bar{Q}\bar{\theta}' - \bar{Q}'\bar{\theta})\,dV + \\ + \gamma\varkappa\int_S (\bar{\theta}'\bar{\theta}_{,n} - \bar{\theta}\bar{\theta}'_{,n})\,dS + \eta\varkappa p\int_B (\bar{T}_{ij}\bar{\varepsilon}'_{ji} - \bar{T}'_{ij}\bar{\varepsilon}_{ij})\,dV.$$

For an infinite body Eq. (3.6) will become much simpler there being no surface integrals. If homogeneous boundary conditions are prescribed on the surface of the body the surface integrals vanish also. Then it can easily be shown that (3.5) splits up into two independent equations: These are:

$$(3.7)\qquad \begin{aligned} &\eta\varkappa p\int_B (\bar{X}_i\bar{u}'_i - \bar{X}'_i\bar{u}_i)\,dV - \gamma\int_B (\bar{Q}\bar{\theta}' - \bar{Q}'\bar{\theta})\,dV = 0, \\ &p\int_B (\bar{T}_{ij}\bar{\varepsilon}'_{ij} - \bar{T}'_{ij}\bar{\varepsilon}_{ij})\,dV = 0. \end{aligned}$$

Similarly, for an infinite space the surface integrals vanish also and (3.6) becomes (3.7) for $B \to V$, where V is the infinite region. The validity of (3.7) follows from the symmetry of operators of the set of Eqs. (2.1) (2.2) and, as a consequence, the symmetry of Green's function. This will be illustrated by means of a simple example.

Let an instantaneous source of heat $Q = \delta(x-\xi)\,\delta(t)$ act at a point ξ of the body, another concentrated source, $Q' = \delta(x-\xi')\,\delta(t)$, acting at ξ'. With no body forces and assuming that $p_i = p_i^* = 0$ and $\theta = \theta' = 0$, we obtain from (3.6):

$$-\gamma \int_B [\delta(x-\xi)\,\overline{\theta}'(x,\xi',p) - \delta(x-\xi')\,\overline{\theta}(x,\xi,p)]\,dV =$$
$$= \eta\varkappa p \int_B [\overline{T}_{ij}(x,\xi,p)\,\overline{\varepsilon}'_{ij}(x,\xi',p) - \overline{T}'_{ij}(x,\xi',p)\,\overline{\varepsilon}_{ij}(x,\xi,p)]\,dV.$$

Hence

$$\overline{\theta}'(\xi,\xi',p) = \overline{\theta}(\xi',\xi,p) - \frac{\eta\varkappa p}{\gamma} \int_B [\overline{T}_{ij}(x_1,\xi,p)\,\overline{\varepsilon}'_{ij}(x,\xi',p) - \overline{T}'_{ij}(x,\xi',p)\,\overline{\varepsilon}_{ij}(x,\xi,p)]\,dV. \tag{3.8}$$

The function $\overline{\theta}(x,\xi,p)$ is easily obtained by solving the set of Eqs. (2.2) and (2.10) with $X_i = 0$ and $Q = \delta(x-\xi)\,\delta(t)$. The matrix of this set of Eqs. being symmetric, it follows that Green's function is symmetric

$$\overline{\theta}(x,\xi,p) = \overline{\theta}(\xi,x,p). \tag{3.9}$$

Therefore, the integral on the right-hand side of the equation (3.8) must be zero which proves our thesis. Analogous considerations may be presented for other perturbations, such as the action of two concentrated impulses of force or one impulse of force and one heat source.

Performing on (3.7) the inverse Laplace transformation, we obtain

$$\eta\varkappa\left\{\int_0^t d\tau \int_B \left[X_i(x,t-\tau)\frac{\partial u'_i(x,\tau)}{\partial\tau} - X'_i(x,t-\tau)\frac{\partial u_i(x,\tau)}{\partial\tau}\right]dV = \right. \tag{3.10}$$
$$= \gamma \int_0^\tau d\tau \left[\int_B Q(x,\tau)\,\theta'(x,t-\tau) - Q'(x,\tau)\,\theta(x,t-\tau)\right]dV,$$

$$\int_0^t d\tau \int_B \left[T_{ij}(x,t-\tau)\frac{\partial \varepsilon'_{ij}(x,\tau)}{\partial\tau} - T'_{ij}(x,t-\tau)\frac{\partial \varepsilon_{ij}(x,\tau)}{\partial\tau}\right]dV. \tag{3.11}$$

Let us observe that the second of the Eqs. (3.7) may be written thus

$$\int_V (\overline{T}_{ij}\overline{\varepsilon}'_{ij} - \overline{T}'_{ij}\overline{\varepsilon}_{ij})\,dV = \int_V \varrho a_0^2 [\overline{e}'\,\overline{\varepsilon}_{33} - \overline{e}\overline{\varepsilon}'_{33} + 2(\overline{\varepsilon}_{13}\,\overline{u}'_{3,1} - \overline{\varepsilon}'_{13}\,\overline{u}_{3,1}) + 2(\overline{\varepsilon}_{3,2}\,\overline{u}_{3,2} - \overline{\varepsilon}'_{3,2}\,\overline{u}_{3,2})]\,dV = 0. \tag{3.12}$$

It is easy to observe that the integral relation will be satisfied locally in the case of the plane problem in the plane normal to the direction of the field H, that is in the x_1x_2-plane, because in this case the derivatives with respect to x_3 vanish. This relation is also satisfied for any one-dimensional problem.

In conclusion of the present section, let us consider in brief the case in which the field vacuum is taken into consideration.

Then, of course, p_i is expressed as

$$p_i = (\sigma_{ij} + T_{ij} - T^*_{ij})\, n_j. \tag{3.13}$$

Eq. (3.6) will be completed, for the vacuum, with equations for Laplace transforms, resulting from (2.6)

$$\int_S [(\vec{E}^{*\prime} x \vec{h}^*) - (\vec{E}^* x \vec{h}^{*\prime})]\, \vec{n}\, dS = 0 \tag{3.14}$$

the normal to the surface is directed towards the interior of the region B_1.

The two sets of Eqs. (3.6) and (3.14) are interrelated by additional relations of continuity of the field (for $\mu \approx 1$)

$$h_n = h_n^*; \quad E_{st} = E_{st}^*, \tag{3.15}$$

where h_n and E_{st} are expressed in an explicit manner in terms of $\vec{u}$ by means of Eqs. (2.3).

In practical problems the field in vacuum is usually negligible and Eqs. (3.6) may be used.

A number of practical formulae can be deduced from the above theorem. In particular the Somigliana theorem can be generalized to the magneto-thermo-elastic problem. The theorem just proved can also be made use of for the obtainment of integral equations of certain boundary-value problems. In a number of cases the computation can be simplified by making use of the second of Eqs. (3.7). The possibilities of application being numerous, it is impossible to discuss them in detail. Let us observe, finally, that considerations analogous to those of the present paper may be presented for real conductors. This will be the object of a separate communication. The case of perfect conductors deserves special consideration, due to its particular simplicity and, in this connection, the practical applicability of the equations obtained.

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С. КАЛИСКИЙ и В. НОВАЦКИЙ, **ТЕОРЕМА ВЗАИМНОСТИ ДЛЯ ПРОБЛЕМЫ МАГНИТО-ТЕРМО-УПРУГОСТИ.**

В работе выведена теорема взаимности для проблемы магнито-термо-упругости в случае идеальных упругих проводников в магнитном поле. Получен ряд практически важных формул, при помощи которых можно конструировать интегральное уравнение краевых проблем магнито-термо-упругости, получить расширение теоремы Сомиглиано, наконец, в ряде особых случаев, упростить вычисления, опираясь на теорему взаимности. Для некоторого особого класса случаев получена теорема с частичным соотношением взаимности (6.7), выраженным лишь тензором Максвелла и вектором перемещения.

В настоящей работе авторы ограничились лишь случаем идеальных проводников ради простоты и эффективности формул. Случай реальных проводников будет рассмотрен отдельно.