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Nr. 7112
Politechnika Warszawska

BULLETIN
DE
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VOLUME XII
NUMÉRO 6

VARSOVIE 1964

Green Functions for an Thermoelastic Medium. I

by

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Presented on March 24, 1964

The aim of the present paper is to give basic solutions of wave equations in an unlimited thermoelastic medium. To present in particular, in a closed form the wave functions as well as the displacements and temperature field formed in an unlimited space under the action of a concentrated force changing harmonically in time.

Let us consider the system of linearized equations of thermoelasticity [1], [2]

$$(1) \quad \nabla^2 \theta - \frac{1}{\kappa} \frac{\partial \theta}{\partial t} - \eta \frac{\partial}{\partial t} \operatorname{div} \mathbf{u} = -\frac{Q}{\kappa},$$

$$(2) \quad \mu \nabla^2 \mathbf{u} + (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{u} + \mathbf{X} = \gamma \operatorname{grad} \theta + \varrho \frac{\partial^2 \mathbf{u}}{\partial t^2}.$$

The first equation is an expanded equation of heat conduction, while the second — the displacement equation (equation of motion) of the theory of elasticity. The equations are mutually coupled. The following notations are adopted: \mathbf{u} denotes the displacement vector, \mathbf{X} — the vector of body forces, $\theta = T - T_0$ — the difference between the absolute temperature T and the temperature characterizing the natural state of heat of the body, T_0 ; Q stands for the function describing the intensity of heat sources. μ and λ are Lamé coefficients with reference to the isothermal state; $\kappa = \lambda_0 / \varrho c$ is a coefficient wherein λ_0 denotes the heat conductivity constant, ϱ — density and c — specific heat, the deformation being assumed constant. Further, $\eta = \gamma T_0 / \lambda_0$, where $\gamma = (3\lambda + 2\mu)$, a_t , a_t being the coefficient of linear heat dilatation. At least, $Q = W / \varrho c$, where W denotes the quantity of heat generated in a volume unit of the body in a time unit. The functions \mathbf{u} , θ , \mathbf{X} , Q are functions of position and time.

Decomposing the displacement vector and the body forces vector into the potential vector and the solenoidal vector

$$(3) \quad \mathbf{u} = \operatorname{grad} \varphi + \operatorname{rot} \boldsymbol{\psi},$$

$$(4) \quad \mathbf{X} = \varrho (\operatorname{grad} \vartheta + \operatorname{rot} \boldsymbol{\chi}),$$

we reduce the system of Eqs. (1) and (2) to system of the following three equations:

$$(5) \quad \left(\nabla^2 - \frac{1}{\varkappa} \partial_t \right) \theta - \eta \partial_t \nabla^2 \varphi = - \frac{Q}{\varkappa},$$

$$(6) \quad \left(\nabla^2 - \frac{1}{c_1^2} \partial_t^2 \right) \varphi = m\theta - \frac{1}{c_1^2} \vartheta,$$

$$(7) \quad \left(\nabla^2 - \frac{1}{c_2^2} \partial_t^2 \right) \Psi = - \frac{1}{c_2^2} \chi, \quad c_1^2 = \frac{\lambda+2\mu}{\varrho}, \quad c_2^2 = \frac{\mu}{\varrho}, \quad \partial_t = \frac{\partial}{\partial t}.$$

After eliminating the temperature θ from Eqs. (5) and (6) we arrive at two wave equations

$$(8) \quad \left(\nabla^2 - \frac{1}{\varkappa} \partial_t \right) \left(\nabla^2 - \frac{1}{c_1^2} \partial_t^2 \right) \varphi - \frac{\varepsilon}{\varkappa} \partial_t \nabla^2 \varphi = - \frac{mQ}{\varkappa} - \frac{1}{c_1^2} \left(\nabla^2 - \frac{1}{\varkappa} \partial_t \right) \vartheta,$$

$$(9) \quad \left(\nabla^2 - \frac{1}{c_2^2} \partial_t^2 \right) \Psi = - \frac{1}{c_2^2} \chi.$$

Eq. (8) is characteristic of the propagation of the longitudinal wave, whereas Eq. (9) — of the transverse wave.

Let us remark that in an unlimited space the heat source and $X' = \varrho \text{ grad } \vartheta$ generate only longitudinal, dilatational waves, while $X'' = \varrho \text{ rot } - \chi$ only transverse waves.

We shall assume that the causes provoking the wave disturbances, namely the heat sources and body forces, change harmonically in time

$$(10) \quad Q(x, t) = Q^*(x) e^{i\omega t}, \quad \vartheta(x, t) = \vartheta^*(x) e^{i\omega t}, \quad \chi(x) = \chi^*(x) e^{i\omega t}.$$

Consequently, the temperature, the displacements and the strains will change harmonically in time, too. Introducing the notations

$$(11) \quad \theta(x, t) = \theta^*(x) e^{i\omega t}, \quad \varphi(x, t) = \varphi^*(x) e^{i\omega t}, \quad \text{etc.}$$

we reduce Eqs. (8) and (9) to the forms

$$(12) \quad (\nabla^2 - k_1^2) (\nabla^2 - k_2^2) \varphi^* = - \frac{mQ^*}{\varkappa} - \frac{1}{c_1^2} (\nabla^2 - q) \vartheta^*,$$

$$(13) \quad (\nabla^2 + \tau^2) \Psi^* = - \frac{1}{c_2^2} \chi^*,$$

where

$$k_1^2 + k_2^2 = q(1 + \varepsilon) - \sigma^2, \quad k_1^2 k_2^2 = -q\sigma^2,$$

$$q = \frac{i\omega}{\varkappa}, \quad \sigma^2 = \frac{\omega^2}{c_1^2}, \quad \tau^2 = \frac{\omega^2}{c_2^2}.$$

k_1^2 and k_2^2 are the roots of the following biquadratic equation

$$k^4 + k^2 [\sigma^2 - q(1 + \varepsilon)] - \sigma^2 q = 0.$$

We shall now consider the action of the concentrated force in an unlimited medium and construct the components of the Green displacement vector. Let us put first that the concentrated unit force — changing harmonically in time — acts at the origin of the coordinate system along the x_1 axis.

In a general approach, for an arbitrary vector of body forces, we determine the functions ϑ^* and χ^* from the formulae [3]

$$(14) \quad \vartheta^*(\mathbf{x}) = -\frac{1}{4\pi\varrho} \int_{(B)} \int \int \left[X_1^*(\mathbf{x}') \frac{\partial}{\partial x_1} \left(\frac{1}{R(\mathbf{x}, \mathbf{x}')} \right) + X_2^*(\mathbf{x}') \frac{\partial}{\partial x_2} \left(\frac{1}{R(\mathbf{x}, \mathbf{x}')} \right) + X_3^*(\mathbf{x}') \frac{\partial}{\partial x_3} \left(\frac{1}{R(\mathbf{x}, \mathbf{x}')} \right) \right] dV(\mathbf{x}'),$$

$$(15) \quad \chi^*(\mathbf{x}) = -\frac{1}{4\pi\varrho} \int_{(B)} \int \int \left\{ i \left[X_2^*(\mathbf{x}') \frac{\partial}{\partial x_3} \left(\frac{1}{R(\mathbf{x}, \mathbf{x}')} \right) - X_3^*(\mathbf{x}') \frac{\partial}{\partial x_2} \left(\frac{1}{R(\mathbf{x}, \mathbf{x}')} \right) \right] + j \left[X_3^*(\mathbf{x}') \frac{\partial}{\partial x_1} \left(\frac{1}{R(\mathbf{x}, \mathbf{x}')} \right) - X_1^*(\mathbf{x}') \frac{\partial}{\partial x_3} \left(\frac{1}{R(\mathbf{x}, \mathbf{x}')} \right) \right] + k \left[X_1^*(\mathbf{x}') \frac{\partial}{\partial x_2} \left(\frac{1}{R(\mathbf{x}, \mathbf{x}')} \right) - X_2^*(\mathbf{x}') \frac{\partial}{\partial x_1} \left(\frac{1}{R(\mathbf{x}, \mathbf{x}')} \right) \right] \right\} dV(\mathbf{x}').$$

Introducing into Eqs. (14) and (15) the expression

$$X_j^*(\mathbf{x}') = \delta(x'_1) \delta(x'_2) \delta(x'_3) \delta_{ij}, \quad j = 1, 2, 3,$$

which characterizes the action of the concentrated force at the origin of the coordinate system along the x_1 axis, we obtain successively:

$$(16) \quad \vartheta^*(\mathbf{x}) = -\frac{1}{4\pi\varrho} \frac{\partial}{\partial x_1} \left(\frac{1}{R} \right), \quad \chi_1^* = 0, \\ \chi_2^* = \frac{1}{4\pi\varrho} \frac{\partial}{\partial x_3} \left(\frac{1}{R} \right), \quad \chi_3^* = -\frac{1}{4\pi\varrho} \frac{\partial}{\partial x_2} \left(\frac{1}{R} \right), \quad R = (x_1^2 + x_2^2 + x_3^2)^{1/2}.$$

Thus, we have to solve the following equations

$$(17) \quad (\nabla^2 - k_1^2) (\nabla^2 - k_2^2) \varphi^* = \frac{1}{4\pi\varrho c_1^2} (\nabla^2 - q) \frac{\partial}{\partial x_1} \left(\frac{1}{R} \right), \\ (\nabla^2 + \tau^2) \psi_2^* = 0, \\ (\nabla^2 + \tau^2) \psi_2^* = -\frac{1}{4\pi\varrho c_2^2} \frac{\partial}{\partial x_3} \left(\frac{1}{R} \right), \\ (\nabla^2 + \tau^2) \psi_3^* = \frac{1}{4\pi\varrho c_2^2} \frac{\partial}{\partial x_2} \left(\frac{1}{R} \right).$$

We shall devote our attention to the first equation of group (17), as the solutions of the remaining equations are known and may be written in the following form:

$$(18) \quad \varphi^* = 0, \quad \psi_2^* = \frac{1}{4\pi\varrho \omega^2} \frac{\partial}{\partial x_3} F_0(R, \omega), \quad \psi_3^* = -\frac{1}{4\pi\varrho \omega^2} \frac{\partial}{\partial x_2} F_0(R, \omega).$$

where

$$F_0(R, \omega) = \frac{e^{-i\omega R}}{R} - \frac{1}{R}.$$

When solving the first equation of the (17) group we take advantage of two properties of the function φ^* . The concentrated force acting along the x_1 axis the wave functions, as well as the displacements, will be axi-symmetric with respect to the x_1 axis. Moreover, the φ^* function is antisymmetric with respect to the $x_2 x_3$ plane, as the displacement u_1 is symmetric with respect to this plane. Considering φ^* as a function of x_1 and of the radius $r = (x_2^2 + x_3^2)^{1/2}$, we perform on the wave equation first the Hankel transformation and then the sine Fourier transformation. Introducing the transformation

$$(19) \quad \tilde{\varphi}(a, \beta) = \sqrt{\frac{2}{\pi}} \int_0^\infty \int_0^\infty \varphi^*(x_1, r) r J_0(ar) \sin \beta x_1 dr dx_1,$$

we reduce the wave equation to the form

$$(20) \quad \tilde{\varphi}(a, \beta) = \frac{1}{4\pi\omega} \sqrt{\frac{2}{\pi}} \left[A_2 \frac{\beta}{a^2 + \beta^2 + k_2^2} - A_1 \frac{\beta}{a^2 + \beta^2 + k_1^2} - \frac{\beta}{a^2 + \beta^2} \right],$$

where

$$A_2 = \frac{(k_2^2 - q) \omega^2}{c_1^2 k_2^2 (k_1^2 - k_2^2)}, \quad A_1 = \frac{(k_1^2 - q) \omega^2}{c_1^2 k_1^2 (k_1^2 - k_2^2)}.$$

Subjecting the relation (20) to the inverse Hankel-Fourier transformation

$$(21) \quad \varphi^*(x_1, r) = \sqrt{\frac{2}{\pi}} \int_0^\infty \int_0^\infty \tilde{\varphi}(a, \beta) a J_0(ar) \sin \beta x_1 da d\beta,$$

we obtain for the function $\varphi^*(x_1, r)$ the following solution in closed form

$$(22) \quad \varphi^*(x_1, r) = -\frac{1}{4\pi\omega} \frac{\partial}{\partial x_1} F(R, \omega),$$

where

$$F = A_2 I_2(R, \omega) - A_1 I_1(R, \omega) - I_0(R, \omega),$$

$$I_j(R, \omega) = \frac{1}{R} e^{-k_j R} \quad j = 1, 2; \quad I_0 = \frac{1}{R}.$$

Notice that passing from the coupled to the non-coupled problem, i.e., putting in the expression for φ^* the following values: $\eta = 0$, $\varepsilon = 0$, $k_1^2 = q$, $k_2^2 = -\sigma^2$, we obtain the known solution of the wave equation of classical elastokinetics

$$(23) \quad \varphi^*(r, x_1) = -\frac{1}{4\pi\omega} \frac{\partial}{\partial x_1} \left(\frac{e^{-i\sigma R}}{R} - \frac{1}{R} \right).$$

Let us return to Eq. (22) and transpose the concentrated force from the origin of the coordinate system to the point (ξ) . Let us, moreover, attach to the functions φ^* and ψ^* indices (1) in order to indicate that they are connected with the force acting along the x_1 axis. We obtain then

$$(24) \quad \begin{aligned} \varphi^{*(1)}(x, \xi) &= -\frac{1}{4\pi\varrho\omega^2} \frac{\partial}{\partial x_1} F(R, \omega), \\ \psi_1^{*(1)}(x, \xi) &= 0, \quad \psi_2^{*(1)} = \frac{1}{4\pi\varrho\omega^2} \frac{\partial}{\partial x_3} F_0(R, \omega), \\ \psi_3^{*(1)} &= -\frac{1}{4\pi\varrho\omega^2} \frac{\partial}{\partial x_2} F_0(R, \omega), \end{aligned}$$

where

$$R = [(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2]^{1/2}.$$

Superposing the displacements conformly to formula (3), we arrive at

$$(25) \quad u_j^{*(1)} = -\frac{1}{4\pi\varrho\omega^2} \{ \partial_1 \partial_j [F(R, \omega) - F_0(R, \omega)] \} + \frac{1}{4\pi\varrho c_2^2} \delta_{ij} \frac{e^{-i\tau R}}{R}, \quad j = 1, 2, 3.$$

The action of the concentrated force in an unlimited medium is accompanied by the temperature field which will be determined from Eq. (6). The amplitude of the temperature field is

$$\theta^{*(1)} = \frac{1}{m} (\nabla^2 + \sigma^2) \varphi^{*(1)} + \frac{1}{c^2 m} \vartheta^{*(1)} = -\frac{1}{4\pi\varrho\omega^2 m} \partial_1 [(\nabla^2 + \sigma^2) F(R, \omega) + \sigma^2 I_0(R)].$$

After some simple calculations we have

$$(26) \quad \theta^{*(1)} = -\frac{qe}{4\pi\varrho m c_1^2 (k_1^2 - k_2^2)} \frac{\partial}{\partial x_1} [I_2(R, \omega) - I_1(R, \omega)].$$

In the non-coupled problem (i.e., for $\varepsilon = 0$) we have $\theta^{*(1)} = 0$.

Returning to the coupled problem of thermoelasticity, we remark that the function $\theta^{*(1)}$ exhibits a singularity at the point $(x) = (\xi)$. Thus, the action of the concentrated force at the point (ξ) generates a concentrated heat source in this point.

Let us direct the concentrated force acting at the point (ξ) subsequently along the x_1 , x_2 and finally along the x_3 axes.

In this way we obtain three groups of components of the displacement vector $u_j^{*(1)}$, $u_j^{*(2)}$, $u_j^{*(3)}$ and the accompanying temperature fields $\theta^{*(1)}$, $\theta^{*(2)}$ and $\theta^{*(3)}$.

The set of all the components of the displacement vector forms the Green displacement tensor

$$(27) \quad G_j^{(k)}(x, \xi, t) = \operatorname{Re} [u_j^{*(k)}(x, \xi, \omega) e^{i\omega t}] = -\frac{1}{4\pi\varrho\omega^2} \operatorname{Re} \left\{ e^{i\omega t} \left[\partial_j \partial_k (F(R, \omega) - F_0(R, \omega)) - \tau^2 \delta_{jk} \frac{e^{-i\tau R}}{R} \right] \right\}.$$



The temperature field $\theta^{*(k)}$ is described by the formula

$$(28) \quad \theta^{(k)}(x, \xi, t) = \operatorname{Re} [\theta^*(x, \xi, \omega) e^{i\omega t}] = -\frac{qe}{4\pi\varrho c_1^2 m (k_1^2 - k_2^2)} \operatorname{Re} \{e^{i\omega t} \partial_k [I_2(R, \omega) - I_1(R, \omega)]\}.$$

Thus, the functions $u_j^{*(k)}$ and $\theta^{*(k)}$ being known, we are able to determine the amplitudes of stresses and strains from the formulae

$$(27) \quad \sigma_{ij}^{*(k)} = 2\mu e_{ij}^{*(k)} + (\lambda e_{ss}^{*(k)} - \gamma \theta^{*(k)}) \delta_{ij},$$

$$(28) \quad \varepsilon_{ij}^{*(k)} = \frac{1}{2} (u_{i,j}^{*(k)} + u_{j,i}^{*(k)}), \quad i, j, s = 1, 2, 3.$$

Now we assume that in the point (ξ) the concentrated unit heat source generates solely the longitudinal waves described by the wave function φ_T^* , which propagate in the form of spherical waves. The solution of Eq. (8) (for $\vartheta = 0$) is known [4]. It is of the form

$$(29) \quad \varphi_T^* = \frac{m}{4\pi\kappa (k_1^2 - k_2^2)} [I_2(R, \omega) - I_1(R, \omega)].$$

We obtain the displacement due to the action of the heat source from the following relation

$$(30) \quad u_j^{*T} = \partial_j \varphi_T^*,$$

wherefrom

$$(31) \quad u_j^{*T} = \partial_j \varphi_T^* = \frac{m}{4\pi\kappa (k_1^2 - k_2^2)} \partial_j [I_2(R, \omega) - I_1(R, \omega)].$$

Let us place the concentrated force acting along the x_j axis in the point (x') . Then the temperature generated by this action in point (ξ') will be obtained from Eq. (26).

$$(32) \quad \theta^{*(j)}(\xi', x', \omega) = -\frac{qe}{4\pi\varrho c_1^2 (k_1^2 - k_2^2)} \partial_j [I_2(R, \omega) - I_1(R, \omega)],$$

$$R = [(x'_1 - \xi'_1)^2 + (x'_2 - \xi'_2)^2 + (x'_3 - \xi'_3)^2]^{1/2}.$$

Now we assume the concentrated unit heat force to be placed in the point (ξ') . The displacement u_j^{*T} (due to the action of this heat source) in point (x') directed along the x_j axis may be obtained from the formula (31).

We have

$$(33) \quad u_j^{*T}(x', \xi', \omega) = \frac{m}{4\pi\kappa (k_1^2 - k_2^2)} \partial_j [I_2(R, \omega) - I_1(R, \omega)].$$

Comparing the formulae (32) and (33), we obtain

$$(34) \quad \theta^{*(j)}(\xi', x', \omega) = -\frac{i\omega\kappa\eta}{mc_1^2 \varrho} u_j^{*T}(x', \xi', \omega).$$

This result should be considered as a corollary to the theorem on the reciprocity for the thermoelastic medium [5]. This theorem may be written — for the unlimited space — under the form:

$$(35) \quad \eta \varkappa i \omega \int \int \int_{(B)} (X_i^* u_i^{*'} - X_i^{*'} u_i^*) dV = \gamma \int \int \int_{(B)} (Q^* \theta^{*'} - Q^{*'} \theta^*) dV.$$

In the case here considered we have

$$X_i^* = \delta(\mathbf{x} - \mathbf{x}') \delta_{ij}, \quad X_i^{*'} = 0, \quad Q^{*'} = \delta(\mathbf{x} - \xi'), \quad Q^* = 0,$$

and Eq. (35) assumes the form

$$\begin{aligned} \eta \varkappa i \omega \int \int \int_{(B)} \delta(\mathbf{x} - \mathbf{x}') \delta_{ij} u_i^{*T}(\mathbf{x}, \xi', \omega) dV(\mathbf{x}) = \\ = -\gamma \int \int \int_{(B)} \delta(\mathbf{x} - \xi') \theta^{*(j)}(\mathbf{x}, \mathbf{x}', \omega) dV(\mathbf{x}), \end{aligned}$$

whence

$$(36) \quad \theta^{*(j)}(\xi', \mathbf{x}', \omega) = -\frac{\eta \varkappa i \omega}{\gamma} u_j^{*T}(\mathbf{x}', \xi', \omega).$$

Taking into account that $\gamma = mc_1^2 \varrho$ we arrive at the conclusion that the formulae (34) and (36) are identical. The theorem on reciprocity leads to two further conclusions, namely:

$$(37) \quad u_j^{*(k)}(\xi', \mathbf{x}', \omega) = u_k^{*(j)}(\mathbf{x}', \xi', \omega),$$

and

$$(38) \quad \theta^{*'}(\mathbf{x}', \xi', \omega) = \theta^*(\xi', \mathbf{x}', \omega).$$

Relation (37) describes the reciprocity of displacements. It means that the displacement $u_j^{*(k)}(\xi', \mathbf{x}', \omega)$ in point (ξ') due to the action of the concentrated unit force applied in point (\mathbf{x}') and acting along the x_k axis is equal to the displacement $u_k^{*(j)}(\mathbf{x}', \xi', \omega)$ in point (\mathbf{x}') along the x_k axis due to the action of the concentrated unit force applied in point (ξ') and acting along the x_j axis.

Relation (38) concerns the reciprocity of temperatures. It means that the unit heat source placed in point (ξ') generates in point (\mathbf{x}') the temperature $\theta^{*'}(\mathbf{x}', \xi', \omega)$, whereas the heat source placed in point (\mathbf{x}') generates heat in point (ξ') its temperature amounting to $\theta^*(\xi', \mathbf{x}', \omega)$.

A more extensive discussion of the problem here considered will be found in a separate paper to be published in the Proceedings of Vibration Problems.

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В. НОВАЦКИЙ, ФУНКЦИИ ГРИНА ДЛЯ ТЕРМОУПРУГОЙ СРЕДЫ. I

В работе дается в замкнутом виде основное решение волновых уравнений в неограниченном термоупругом пространстве. В частности, даются функции перемещения тензора Грина, а также поля температуры, сопутствующие деформациям.

