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APPLIED MECHANICS

The Plane Problem of Magnetothermoelasticity. II

by

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1. Introduction

In [1] the two-dimensional dynamic problem has been considered concerning the propagation of thermoelastic waves in a perfectly conductive infinite medium, subjected to the action of a steady magnetic field.

It has been there assumed that the state of strain, the temperature and magnetic fields, excited by the body forces and heat sources, depend on the space coordinates x_1, x_2 and on the time t only, and, moreover, that the initial steady magnetic field $\vec{H} = (0, 0, H_3)$ is directed along the axis x_3.

In the present contribution a method of solving the fundamental equations of the problem will be given using three auxiliary functions (of the type of Galerkin's functions known from the theory of elasticity), as well as procedure of approximate solving this system of equations. Furthermore, the boundary problem for the medium consisting of the thermoelastic half-space adjoining the vacuum will be formulated.

The fundamental equations of the problem with the corresponding notations are as in [1].

The first group consists of the equations of displacements

$$\mu\nabla^2 \vec{u} + (\lambda+\mu+a_0^2\,\varrho) \operatorname{grad} \operatorname{div} \vec{u} + \vec{F} = \gamma \operatorname{grad} \theta + \varrho \ddot{\vec{u}}, \tag{1.1}$$

$$\vec{u} = (u_1, u_2, 0), \qquad \vec{F} = (F_1, F_2, 0),$$

$$a_0^2 = \frac{H_3^2\,\mu_0}{4\pi\varrho}, \qquad \nabla_1^2 = \partial_1^2 + \partial_2^2.$$

Here, $\vec{u}$ denotes the displacement vector, $\vec{F}$ — the vector of the body forces, θ — the temperature measured starting from the natural state of the body ($\theta = 0$), μ, λ are the isothermic Lamé constants, a_0 — Alfven's velocity, ϱ — the density, and $\gamma = (3\lambda+2\mu)\,\alpha_t$. By α_t we denote the coefficient of thermal linear expansion, and by μ_0 — the constant of magnetic permeability.

The second group constitute the electrodynamic equations of slowly moving, media

$$E_1 = -\frac{\mu_0 H_3}{c}\dot{u}_2, \quad E_2 = \frac{\mu_0 H_3}{c}\dot{u}_1, \quad E_3 = 0, \tag{1.2}$$

$$\dot{h}_1 = 0, \quad \dot{h}_2 = 0, \quad \dot{h}_3 = -\frac{c}{\mu_0}(\partial_1 E_2 - \partial_2 E_1), \tag{1.3}$$

$$j_1 = \frac{c}{4\pi}\partial_2 h_3, \quad j_2 = -\frac{c}{4\pi}\partial_1 h_3, \quad j_3 = 0. \tag{1.4}$$

In the above equations E_i $(i = 1, 2, 3)$ stand for the components of the vector of the electric field intensity $\vec{E}$, h_i $(i = 1, 2, 3)$ — for the components of the magnetic field intensity $\vec{h}$, and j_i $(i = 1, 2, 3)$ — for the components of the current density $\vec{j}$. The light velocity is denoted by c.

The last relation of the system of differential equations is represented by the equation of heat conductivity

$$\nabla_1^2 \theta - \frac{1}{\varkappa}\dot{\theta} - \eta \operatorname{div} \vec{u} = -Q/\varkappa. \tag{1.5}$$

This is the generalized heat conductivity equation, [2], taking into account the coupling of the strain state with the field of temperature. Here $\varkappa$ denotes the heat conductivity, $Q = W/c_\varepsilon$, where W is the quantity of heat created in the volume unit per time unit, c_ε — the specific heat at constant strain, referred to volume unit, $\eta = \gamma T_0/k$, where k denotes the coefficient of heat conductivity, and T_0 the absolute temperature of the body at the natural state (thus, for $\theta = 0$).

Finally, we write the relations connecting the displacements u_i with the strain components ε_{ij}, and the stresses σ_{ij} with the strains ε_{ij} and the temperature θ

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad i, j = 1, 2, \tag{1.6}$$

$$\left\{ \begin{array}{l} \sigma_{ij} = 2\mu\, \varepsilon_{ij} + (\lambda \varepsilon_{kk} - \gamma\theta)\, \delta_{ij}, \quad i, j = 1, 2, \\ \sigma_{33} = \lambda \varepsilon_{kk} - \gamma\theta. \end{array} \right. \tag{1.7}$$

Relations (1.7) are known from thermoelasticity as the Duhamel-Neumann equations.

2. Solving functions φ_i $(i = 1, 2, 3)$

Eqs. (1.1) and (1.5) are coupled with each other. Dividing (1.1) by μ they can be represented in the form:

$$\begin{aligned} &(\Box_2^2 + \beta\partial_1^2)\, u_1 + \beta\partial_1\partial_2\, u_2 - m_0\, \partial_1\, \theta = -X_1, \\ &\beta\partial_1\partial_2\, u_1 + (\Box_2^2 + \beta\partial_2^2)\, u_2 - m_0\, \partial_2\, \theta = -X_2, \\ &-\eta\partial_1\, \partial_t u_1 - \eta\partial_2\, \partial_t\, u_2 + \Box_3^2\, \theta = -X_3. \end{aligned} \tag{2.1}$$

where the following notations have been introduced

$$\gamma/\mu = m_0, \quad \frac{\mu}{\varrho} = c_2^2, \quad \lambda+\mu+a_0^2\,\varrho = \beta,$$

(2.2) $$\Box_2^2 = \nabla_1^2 - \frac{1}{c_2^2}\partial_t^2, \quad \Box_3^2 = \nabla_1^2 - \frac{1}{\varkappa}\partial_t, \quad \partial_i = \frac{\partial}{\partial x_i}, \quad i = 1, 2,$$

$$\partial_t = \frac{\partial}{\partial t}, \quad X_1 = F_1/\mu, \quad X_2 = F_2/\mu, \quad X_3 = Q/\varkappa.$$

Eqs. (2.1) can also be represented in the operator form

(2.3) $$\mathcal{L}_{ij}(w_j) = -X_i, \quad i, j = 1, 2, 3,$$

where

$$w_1 = u_1, \quad w_2 = u_2, \quad w_3 = \theta,$$

(2.4) $$\mathcal{L}_{11} = \Box_2^2+\beta\partial_1^2, \quad \mathcal{L}_{12} = \beta\partial_1\partial_2 = \beta\partial_2\partial_1, \text{ etc.},$$

and it should be observed that the operators $\mathcal{L}_{13}$ and $\mathcal{L}_{23}$ are non-symmetric.

Let us introduce three functions φ_i $(i = 1, 2, 3)$ connected with the displacements and the temperature by the following relations:

(2.5) $$w_1 = \begin{vmatrix} \varphi_1 & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \varphi_2 & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \varphi_3 & \mathcal{L}_{32} & \mathcal{L}_{33} \end{vmatrix} \quad w_2 = \begin{vmatrix} \mathcal{L}_{11} & \varphi_1 & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \varphi_2 & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \varphi_3 & \mathcal{L}_{33} \end{vmatrix} \quad w_3 = \begin{vmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \varphi_1 \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \varphi_2 \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \varphi_3 \end{vmatrix}$$

or, in another form:

$$w_1 = u_1 = [\Box_3^2(\Box_2^2+\beta\partial_2^2) - m_0\,\eta\partial_t\,\partial_2^2]\,\varphi_1+\partial_1\,\partial_2(\eta m_0\,\partial_t - \beta\,\Box_3^2)\,\varphi_2+ \\ +m_0\,\Box_2^2\,\partial_1\,\varphi_3,$$

(2.6) $$w_2 = u_2 = \partial_1\,\partial_2(\eta m_0\,\partial_t - \beta\,\Box_3^2)\,\varphi_1+[\Box_3^2(\Box_2^2+\beta\partial_1^2) - m_0\,\eta\partial_t\,\partial_1^2]\,\varphi_1+ \\ +m_0\,\Box_2^2\,\partial_2\,\varphi_3,$$

$$w_3 = \theta = \eta\partial_t\,\Box_2^2(\partial_1\,\varphi_1+\partial_2\,\varphi_2)+(\Box_2^2+\beta\nabla_1^2)\,\Box_2^2\,\varphi_3.$$

Substitution of the quantities w_i $(i = 1, 2, 3)$ from formulae (2.6) into Eqs. (2.4) yields the following three non-homogeneous equations determining the functions φ_i $(i = 1, 2, 3)$:

(2.7) $$\Box_2^2\,[(\Box_2^2+\beta\nabla^2)\,\Box_3^2 - m_0\,\eta\partial_t\,\nabla_1^2]\,\varphi_i = -X_i, \quad i = 1, 2, 3.$$

Furthermore, two quantities characterizing the wave propagation, the dilatation $e = \partial_1\,u_1+\partial_2\,u_2$ and the component of rotation along the x_3 axis $\omega_3 = \frac{1}{2}(\partial_1\,u_2 - \partial_2\,u_1)$ can be expressed in terms of functions φ_i

(2.8) $$e = \Box_2^2\,[\Box_3^2(\partial_1\,\varphi_1+\partial_2\,\varphi_2)+m_0\,\nabla_1^2\,\varphi_3],$$

(2.9) $$\omega_3 = \tfrac{1}{2}\,[\Box_3^2(\Box_2^2+\beta\nabla_1^2) - m_0\,\eta\partial_t\,\nabla_1^2]\,(\partial_1\,\varphi_2 - \partial_2\,\varphi_1).$$

The course of the procedure is the following. For given body forces $\vec{F} = (F_1, F_2)$ and heat sources Q, the functions φ_1 are found from Eqs. (2.7). Next, the displacements u_i, the temperature θ and the quantities e and ω_3 are determined from formulae (2.6), (2.7) and (2.9), and then the strains and stresses from formulae (1.6) and (1.7), while the electromagnetic quantities from formulae (1.2)—(1.4).

Let us observe that in the particular case where the heat sources vanish ($Q = 0$) we have $\varphi_3 = 0$. Only two of Eqs. (2.7) are then at our disposal, and formulae (2.6), (2.8), (2.9) simplify considerably.

If no body forces act ($F_1 = 0$, $F_2 = 0$), then we have $\varphi_1 = 0$, $\varphi_2 = 0$. In this case only the third of Eqs. (2.7) holds. Introducing the notation $\zeta = m_0 \Box_2^2 \varphi_3$, this equation can be represented in the form

$$(\Box_1^2 \Box_3^2 - \eta m \partial_t \nabla_1^2)\, \zeta = -\frac{mQ}{\varkappa}, \tag{2.10}$$

where

$$m = \frac{\gamma}{a^2 \varrho}, \quad a^2 \varrho = \lambda + 2\mu + a_0^2 \varrho, \quad \Box_1^2 = \nabla^2 - \frac{1}{a^2} \partial_t^2 .$$

In this particular case we have

$$u_1 = \partial_1 \zeta, \quad u_2 = \partial_2 \zeta, \quad \theta = \frac{1}{m} \Box_1^2 \zeta, \tag{2.11}$$

and

$$e = \nabla^2 \zeta, \quad \omega_3 = 0. \tag{2.12}$$

Here we deal with the longitudinal wave only, and it follows from the nature of Eq. (2.10) that this wave undergoes damping and dispersion. Function ζ and Eq. (2.10) coincide with function Φ and Eq. (2.18) from paper [1], which has been derived in another way by resolving the displacement vector into the potential and rotational parts.

The procedure presented in this section is important for the cases where the body forces cannot be decomposed into the potential and rotational parts (e.g., in the case of concentrated forces).

3. Approximate solutions

The exact solution of the conductivity equations in terms of displacements coupled with the heat conductivity equation encounters considerable mathematical difficulties. However, it is known from thermoelasticity that the effect of the coupling of the strain state and the temperature field contributes only to small changes in the values of the displacements and stresses. We may therefore expect a considerable simplification of the solution by using the method of perturbation (method of small parameter).

In [1] it has been shown that by the substitutions

$$u_1 = \partial_1 \Phi - \partial_2 \psi, \quad u_2 = \partial_2 \Phi + \partial_1 \psi, \tag{3.1}$$

$$F_1 = \varrho (\partial_1 \vartheta - \partial_2 \psi), \quad F_2 = \varrho (\partial_2 \vartheta + \partial_1 \chi), \tag{3.2}$$

the system of relations (1.1) and (1.5) can be reduced to the following system of equations

$$\Box_1^2 \Phi + \frac{1}{a^2} \vartheta = m\theta, \tag{3.3}$$

$$\Box_2^2 \psi + \frac{1}{c_2^2} \chi = 0, \tag{3.4}$$

$$\Box_3^2 \theta - \eta \nabla^2 \dot{\Phi} = -Q/\varkappa. \tag{3.5}$$

Eliminating the temperature θ from Eqs. (3.3) and (3.5), we obtain

$$\left(\Box_1^2 \Box_3^2 - \frac{\varepsilon}{\varkappa} \partial_t \nabla_1^2\right) \Phi = -\frac{mQ}{\varkappa} - \frac{1}{a^2} \Box_3^2 \vartheta, \qquad \varepsilon = \eta m \varkappa. \tag{3.6}$$

The most difficulties are connected with the solution of Eq. (3.6). In this equation the quantity ε can be assumed as a very small one, [3], ($\varepsilon = 1.68 \cdot 10^{-2}$ for copper, $\varepsilon = 2.97 \cdot 10^{-4}$ for steel, $\varepsilon = 7.33 \cdot 10^{-2}$ for lead) and the function Φ can be expanded into an infinite series of the powers of ε

$$\Phi = \sum_{r=0}^{\infty} \Phi^{(r)} \varepsilon^r. \tag{3.7}$$

Inserting (3.7) into (3.6), we arrive at the following system of equations

$$\Box_1^2 \Box_3^2 \Phi^{(0)} = -\frac{Q}{\varkappa} - \frac{1}{a^2} \Box_3^2 \vartheta, \tag{3.8}$$

$$\Box_1^2 \Box_3^2 \Phi^{(r)} = \frac{1}{\varkappa} \partial_t \nabla_1^2 \Phi^{(r-1)}. \tag{3.9}$$

Eq. (3.8) can be readily solved. It can be decomposed into the wave equation

$$\Box_1^2 \Phi^{(0)} + \frac{1}{a^2} \vartheta = m\theta^{(0)}, \tag{3.10}$$

and the classical equation of heat conductivity

$$\Box_3^2 \theta^{(0)} = -Q/\varkappa. \tag{3.11}$$

Eq. (3.10) determines the longitudinal wave which undergoes neither damping nor dispersion and propagates with the velocity a. Function $\Phi^{(0)}$ being known, the function $\Phi^{(0)}$ can be obtained from Eq. (3.9). Using again Eq. (3.9), we determine successively the functions $\Phi^{(2)}$, $\Phi^{(3)}$, etc.

In technological computations it suffices to retain the second term of the series (3.7) only.

The first approximation can be regarded as the solution of the problem where the coupling of the electromagnetic field with the temperature field has been taken into account, while the coupling of the strain state with the field of temperature has been neglected.

An analogous procedure can be applied to the system of Eqs. (2.7) and relations (2.6) by assuming

$$\varphi_i = \sum_{r=0}^{\infty} \varphi_i^{(r)} \varepsilon^r, \qquad w_i = \sum_{r=0}^{\infty} w_i^{(r)} \varepsilon^r, \qquad i = 1, 2, 3. \tag{3.12}$$

For the first approximation we obtain the system of equations

$$\Box_1^2 \Box_2^2 \Box_3^2 \varphi_i^{(0)} = -\frac{X_i \mu}{\varrho a^2}, \qquad i = 1, 2, 3. \tag{3.13}$$

4. Elastic half-space adjoining the vacuum

Let us consider the thermoelastic half-space $x_1 \geqslant 0$ adjoining the vacuum $x_1 \leqslant 0$. We assume that both in the thermoelastic half-space and in the vacuum there exists an initial steady electromagnetic field $\vec{H} = (0, 0, H_3)$.

The thermoelastic half-space is described by Eqs. (1.1)—(1.7), while for the vacuum the following wave equations and relations are valid

$$\left(\nabla_1^2 - \frac{1}{c^2}\partial_t^2\right)\vec{E}^* = 0, \qquad \left(\nabla_1^2 - \frac{1}{c^2}\partial_t^2\right)\vec{h}^* = 0, \tag{4.1}$$

$$\operatorname{rot}\vec{E}^* = -\frac{1}{c}\dot{\vec{h}}^*, \qquad \operatorname{rot}\vec{h}^* = \frac{1}{c}\dot{\vec{E}}^*. \tag{4.2}$$

We have denoted the vectors of the electric and electromagnetic field intensities by $\vec{E}^*$ and $\vec{h}^*$, respectively. For the two-dimensional problems considered we have

$$\dot{E}_1^* = c\partial_2 h_3^*, \quad \dot{E}_2^* = -c\partial_1 h_3^*, \quad \dot{E}_3^* = 0, \tag{4.3}$$

$$\dot{h}_1^* = 0, \quad \dot{h}_2^* = 0, \quad \dot{h}_3^* = -c(\partial_1 E_2^* - \partial_2 E_1^*). \tag{4.4}$$

By virtue of (4.3) and (4.4) relation (4.1) reduces to three wave equations

$$\mathcal{D}_1 E_1^* = 0, \quad \mathcal{D}E_2^* = 0, \quad \mathcal{D}h_3^* = 0, \quad \mathcal{D} = \nabla_1^2 - \frac{1}{c^2}\partial_t^2. \tag{4.5}$$

For the given vertical and horizontal loadings $p_1(x_2, t)$, $p_2(x_1, t)$ and the temperature $\theta(0, x_2, t)$, we have at the plane between the thermoelastic medium and the vacuum the following boundary conditions, [4], [5],

$$\sigma_{11}(0, x_2, t) + T_{11}(0, x_2, t) - T_{11}^*(0, x_2, t) + p_1(x_2, t) = 0, \tag{4.6}$$

$$\sigma_{12}(0, x_2, t) + T_{12}(0, x_2, t) - T_{12}^*(0, x_2, t) + p_2(x_2, t) = 0, \tag{4.7}$$

$$\theta(0, x_2, t) = f(x_2, t), \tag{4.8}$$

$$E_1(0, x_2, t) = E_1^*(0, x_2, t), \quad E_2(0, x_2, t) = E_2^*(0, x_2, t), \quad h_3(0, x_2, t) = h_3^*(0, x_2, t). \tag{4.9}$$

Eqs. (4.6) and (4.7) represent the effect of the vertical and horizontal loadings acting in the plane $x_1 x_3$, while T_{11}, T_{12} and T_{11}, T_{12} are the components of Maxwell's potential tensor for the solid body and the vacuum, respectively. Condition (4.9) determines the distribution of the temperature at the boundary $x_1 = 0$. Finally, relations (4.9) are the conditions of continuity of the electromagnetic quantities in the plane of contact.

Maxwell's potential tensors are given by formulae

$$T_{ij} = \frac{1}{4\pi}(H_i h_j + H_j h_i - \delta_{ij} \vec{H}\vec{h}), \tag{4.10}$$

$$T_{ij}^* = \frac{1}{4\pi}(H_i h_j^* + H_j h_i^* - \delta_{ij} \vec{H}\vec{h}^*), \qquad i, j = 1, 2, 3 \tag{4.11}$$

Taking into account relations (1.2), (1.3) and (4.3), (4.4), we obtain from Eqs. (4.10) and (4.11)

$$T_{11} = -\frac{\mu_0}{4\pi} H_3 h_3, \qquad T_{11}^* = -\frac{\mu_0}{4\pi} H_3 h_3^*. \tag{4.12}$$

Bearing in mind relations (1.6), (1.7) and (1.2), (1.3), we have

$$\begin{aligned} \sigma_{11} &= 2\mu\, \partial_1 u_1 + \lambda(\partial_1 u_1 + \partial_2 u_2) - \gamma\theta, \\ \sigma_{12} &= \mu(\partial_1 u_2 + \partial_2 u_1), \\ h_3 &= -H(\partial_1 u_1 + \partial_2 u_2). \end{aligned} \tag{4.13}$$

By virtue of (4.12) and (4.13) the boundary conditions (4.6)—(4.9) at the plane $x_1 = 0$ can be represented in the following form:

$$\partial_1 u_1 + \frac{a_1^2}{a^2}\partial_2 u_2 - m\theta + \frac{p_1}{\varrho a^2} + \frac{\mu_0}{4\pi\varrho a^2} H_3 h_3^* = 0, \qquad a_1^2 = \lambda/\mu + a_0^2, \tag{4.6'}$$

$$\partial_1 u_2 + \partial_1 u_1 + p_2 = 0, \tag{4.7'}$$

$$\theta = f, \tag{4.8'}$$

$$-\frac{\mu_0 H_3}{c^2}\ddot{u}_2 - \partial_2 h_3^* = 0, \qquad \frac{\mu_0 H_3}{c^2}\ddot{u}_1 + \partial_1 h_3^* = 0, \tag{4.9'}$$

$$h_3 = h_3^* = -H_3(\partial_1 u_1 + \partial_2 u_2).$$

It can readily be seen that the number of boundary conditions is equal to that of equations.

For the one-dimensional problem, where all quantities depend on x_1 and t only, the boundary conditions simplify considerably.

The relations presented in this section show that the loading or heating of plane $x_1 = 0$ and also the action of body forces and heat sources within the thermoelastic space is inseparably connected with the excitation of electromagnetic waves in the vacuum. The problem simplifies considerably by disregarding the coupling of the temperature field with the strain state ($\eta = 0$). The one-dimensional problem concerning the sudden heating of plane $x_1 = 0$ has been discussed in detail in papers [6] and [7].

A further generalization of the present paper to cover two-dimensional problems (propagation of Rayleigh's waves, wave propagation in a layer, etc.), and one-dimensional problems, will be published in Archiwum Mechaniki Stosowanej in 1963.

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В. НОВАЦКИЙ, **ПЛОСКАЯ ЗАДАЧА МАГНИТОТЕРМОУПРУГОСТИ. II**

В настоящей работе, которая является продолжением работы [1], показан метод решения плоской сопряженной задачи магнитотермоупругости в неограниченном пространстве при использовании трех решающих функций φ_i ($i = 1, 2, 3$). В частном случае воздействия единственно источников тепла, для решения задачи достаточна одна только функция $\varphi_3 = \varphi$.

В разделе 3 работы дается метод приближенного решения основных волновых уравнений при применении пертурбационного метода. Наконец — в разделе 4 приводятся системы уравнений и краевые условия для случая термоупругого полупространства, находящегося в контакте с вакуумом.