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APPLIED MECHANICS (ELASTICITY)

Formulation of a Boundary Problem of the Theory of Elasticity with Mixed Boundary Conditions

by

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In a previous paper [1] we formulated the static boundary problem with mixed boundary conditions leading to a system of integral equations. We made profit in our considerations of a method which may be called an extension of the method of "forces" and that of "deformations" used in construction statics.

In the present communication — basing on the general theorem on the reciprocity of displacements — a formulation of boundary problem of elastostatics is given using Green's functions elaborated for a more simple problem.

We assume as our starting point E. Betti's theorem on reciprocity*)

$$\int_B (F_i v_i' - F_i' v_i)\, dV + \int_S (p_i v_i' - p_i' v_i)\, d\sigma = 0\,. \tag{1}$$

We have here two systems of loads acting upon the body. In the first one external loads p_i and mass forces F_i are acting, resulting in displacements v_i, deformations ε_{ij} and stress σ_{ij}. In the second system, the loads p_i', F_i' are responsible for displacement v_i' and for the components of the deformation (ε_{ij}') and stress (σ_{ij}') states.

Let us consider a simply connected elastic body with region B and a surface S, fully clamped on the surfaces S_1 and S_3, being free on the surface S_2. We denote by $\vec{q}$ the forces acting on the surface S_2, and the mass forces by $\vec{X}$ (Fig. 1).

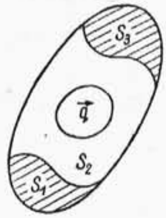

Fig. 1

Due to the load on the region S_3 reactions $\vec{R}$ will appear; we shall consider them as unknown functions of our problem. If we assume $\vec{q} = 0$, $\vec{R} = 0$, then we obtain

*) Index-tensor notation is used in Eq. (1) as well as in following ones.

an initial system — called basic system — an elastic body fully clamped on the surface S_1 (Fig. 2).

We assume that in this system displacement and deformation, due to every arbitrary load, may be determined.

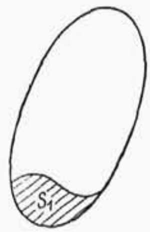

Fig. 2

Let us now choose two systems of loads. The first one refers to the load presented in Fig. 1. In this system the displacements $\vec{u}$, the stress state σ_{ij} and the deformation ε_{ij} are due to the load $\vec{q}$ and the mass forces $\vec{X}$.

The displacements shall satisfy the equations

$$\mu u_{i,jj} + (\lambda+\mu) u_{j,ji} + X_i = 0 \tag{2}$$

with boundary conditions

$$u_i = 0 \quad \text{on} \quad S_1 \quad \text{and} \quad S_3; \quad \sigma_{ij} n_j = q_i \quad \text{on} \quad S_2 \quad \text{and} \quad \sigma_{ij} n_j = R_i \quad \text{on} \quad S_3. \tag{3}$$

Let us choose the second system of loads to be an action of concentrated unit mass force applied at the point $Q \equiv (\xi_1, \xi_2, \xi_3)$ and directed along the x_k-axis in the basic system. This load will cause in the basic system the displacement $\vec{G}^{(k)}$ with the components $G_i^{(k)}$ $(i = 1, 2, 3)$. The displacements $G_i^{(k)}$ will satisfy the following equation

$$\mu\, G_{i,jj}^{(k)} + (\lambda+\mu)\, G_{j,ji}^{(k)} + \delta(x-\xi)\,\delta_{ik} = 0 \qquad (i, k = 1, 2, 3) \tag{4}$$

with boundary conditions:

$$G_i^{(k)} = 0 \quad \text{on} \quad S_1; \quad \tau_{ij}^{(k)} n_j = 0 \quad \text{on} \quad S_2 \quad \text{and} \quad S_3. \tag{5}$$

In Eqs. (4) we have: $\delta(x-\xi) = \delta(x_1-\xi_1)\,\delta(x_2-\xi_2)\,\delta(x_3-\xi_3)$.

We denoted here by $\tau_{ij}^{(k)}$ the stress tensor, due to the action of a concentrated force applied at the point Q and directed along the x_k-axis. Eqs. (4) present a system of 9 equations, 3 displacement equations for each value $k = 1, 2, 3$.

Solving Eqs. (4), we obtain 9 functions $G_i^{(k)}$; $G_i^{(k)}(x, \xi) = G_k^{(i)}(\xi, x)$.

We shall take advantage of Eq. (1) with due regard to both systems of loads considered, $v_i = u_i$, $v_i' = G_i^{(k)}$, ... We have then

$$\int_B [X_i(x)\, G_i^{(k)}(x, \xi) - \delta(x-\xi)\,\delta_{ik}\, u_i(x)]\, dV(x) + \int_{S_2} q_i(x)\, G_i^{(k)}(x, \xi)\, d\sigma(x) + \int_{S_3} R_i(x)\, G_i^{(k)}(x, \xi)\, d\sigma(x) = 0, \tag{6}$$

wherefrom we obtain

$$u_k(\xi) = \int_V X_i(x)\, G_i^{(k)}(x, \xi)\, dV(x) + \int_{S_2} q_i(x)\, G_i^{(k)}(x, \xi)\, d\sigma(x) + \int_{S_3} R_i(x)\, G_i^{(k)}(x, \xi)\, d\sigma(x). \tag{7}$$

The two first integrals appearing in Eq. (7) may be considered as a displacement $u_k^0(\xi)$, due in the basic system (Fig. 2) to the action of the load $\vec{q}$. The displacements $u_k^0(x)$ should satisfy the following system of equations

$$\mu u^0_{i,jj} + (\lambda+\mu)\, u^0_{j,ji} + X_i = 0, \tag{8}$$

with the boundary conditions

$$u_i^0 = 0 \quad \text{on} \quad S_1; \quad \sigma_{ij}^0 n_j = q_i \quad \text{on} \quad S_2; \quad \sigma_{ij}^0 n_j = 0 \quad \text{on} \quad S_3. \tag{9}$$

Assuming $v_i = u_i^0$, $v_i' = G_i^{(k)}$ and so on, and making use of Eq. (1), we obtain

$$\int_V [X_i(x)\, G_i^{(k)}(x, \xi) - \delta(x-\xi)\, \delta_{ik}\, u_i^0(x)]\, dV(x) + \int_{S_2} q_i(x)\, G_i^{(k)}(x, \xi)\, d\sigma(x) = 0, \tag{10}$$

whence

$$u_k^0(\xi) = \int_V X_i(x)\, G_i^{(k)}(x, \xi)\, dV(x) + \int_{S_2} q_i(x)\, G_i^{(k)}(x, \xi)\, d\sigma(x). \tag{11}$$

Thus, we may write Eq. (7) in the following form

$$u_k(\xi) = u_k^0(\xi) + \int_{S_3} R_i(x)\, G_i^{(k)}(x, \xi)\, d\sigma(x). \tag{12}$$

In the above equation the unknown functions are: $R_i(x)$ on S_3 and $u_k(\xi)$ in the region B.

In order to determine the functions $R_i(x)$, we shall take advantage of the boundary condition stating that an elastic body is fully clamped on the surface S_3. Consequently, it should be

$$u_k(x') = 0 \quad \text{on} \quad S_3 \quad (k = 1, 2, 3), \quad (x') \in S_3. \tag{13}$$

Hence, moving with the point $(\xi) \in B$ to the point (x') on the surface S_3, we obtain from Eq. (12), taking into account the relation (13), the following system of integral equations

$$u_k^0(x') + \int_{S_3} R_i(x) G_i^{(k)}(x, x')\, d\sigma(x) = 0 \quad (x) \in S_3, \quad (x') \in S_3. \tag{14}$$

It is Fredholm's system of three integral equations of the first type. If we determine the functions $R_i(x)$, $i = 1, 2, 3$ from this system of equations, the displacement $u_k(\xi)$ may be obtained from Eq. (12). For the case when the surface S_3 degenerates to a line (full clamping along the line), the surface integral transforms into a curvilinear one.

It may happen that the surface support on S_3 is of a form which allows only surface reactions normal to S_3. Then additional linear relations between R_i components will appear. This in turn will enable us to reduce the number of unknown functions in relation (12) and will, moreover, reduce the system of integral equations (14) to one equation only.

This method may be extended to an elastic body supported by other surfaces $S_4, S_5, \dots$.

Finally, let us observe that Eq. (12) may be written in the form*)

$$u_k(\xi) = u_k^0(\xi) + \int_{S_3} R_i(x)\, U_k^{(i)}(\xi, x)\, d\sigma(x), \tag{15}$$

where $U_k^{(i)}(\xi, x)$ stands for the component of the state of displacement in the point $(\xi) \in B$ in the basic system directed along the x_k-axis and due to the action of a concentrated unit mass force applied at the point $(x) \in S_3$ and directed along the x_i-axis. It follows directly from the J. Maxwell's theorem on reciprocity

$$G_i^{(k)}(x, \xi) = U_k^{(i)}(\xi, x). \tag{16}$$

Taking into account Eq. (16), we may write the system of Eqs. (14) in the following form:

$$u_k^0(x') + \int_{S_3} R_i(x)\, U_k^{(i)}(x', x)\, d\sigma(x) = 0, \quad (k = 1, 2, 3), \quad (x), (x') \in S_3. \tag{17}$$

When solving the problem presented in Fig. 1, we may proceed still in another way. Let us divide the body by the transsection $\alpha - \alpha$ into two regions I and II. Let us assume as unknown functions of our problem (Fig. 3) the forces of mutual interaction $\vec{R}^{(\alpha)}$ of both parts in the cross-section $\alpha - \alpha$. Assuming $\vec{q} = 0$, $\vec{X} = 0$, $\vec{R}^{(\alpha)} = 0$, we obtain two basic systems, the system I being clamped on S_1 and the system II on S_3.

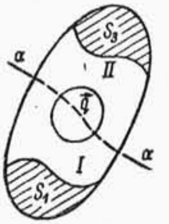

Fig. 3

According to the formula (12), we have in the system I:

$$w_k^{\mathrm{I}}(\xi_{\mathrm{I}}) = w_k^{0,\mathrm{I}}(\xi_{\mathrm{I}}) + \int_{S_\alpha} R_i^{(\alpha)}(x_\alpha)\, G_i^{\mathrm{I}(k)}(x_\alpha, \xi_{\mathrm{I}})\, d\sigma(x_\alpha), \quad (\xi_{\mathrm{I}}) \in B_{\mathrm{I}}, \quad (x_\alpha) \in S_\alpha \tag{18}$$

Similarly, we have in the region II:

$$w_k^{\mathrm{II}}(\xi_{\mathrm{II}}) = w_k^{0,\mathrm{II}}(\xi_{\mathrm{II}}) + \int_{S_\alpha} R_i^{(\alpha)}(x_\alpha)\, G_i^{\mathrm{II}(k)}(x_\alpha\, \xi_{\mathrm{II}})\, d\sigma(x_\alpha), \quad (\xi_{\mathrm{II}}) \in B_{\mathrm{II}}, \quad (x_\alpha) \in S_\alpha. \tag{19}$$

*) It is in this form that the solution of the problem was given in [1]. The formula (17) should read, of course, $R_i U_k^{(i)} = R_1 U_k^{(1)} + R_2 U_k^{(2)} + R_3 U_k^{(3)}$.

We obtain the unknown functions $R_i^{(a)}$ from the conditions stating that the relative displacements of the systems I and II on the surface S_α are equal to zero

$$(20)\quad w_k^{\mathrm{I},0}(x'_a)-w_k^{\mathrm{II},0}(x'_a)+\int_{S_a} R_i^{(a)}(x_\alpha)\,[G_i^{\mathrm{I}(k)}(x_\alpha, x'_a)+G_i^{\mathrm{II}(k)}(x_\alpha, x'_a)]\,d\sigma(x_\alpha)=0,\qquad (x_\alpha),(x'_a)\in S_\alpha .$$

The modus procedendi proposed here is particularly convenient in the case of a body with an aperture (Fig. 4). Then in Eqs. (18)÷(20) the integration should be carried out on the region S'_a, where $S'_a = S_\alpha - S_s$, S_s being the surface of the aperture.

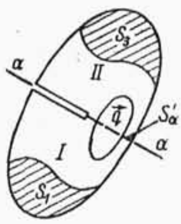

Fig. 4

Up till now we proceeded in a way similar to that called the method of forces used in construction statics, where support reactions, support moments and resultant inner forces are considered as unknown.

Nevertheless, we may proceed in a different manner, assuming the functions of displacement as unknown, for instance, as in the case presented in Fig. 1 the displacements $u_i(x)$ $(x)\in S_2$ on the surface S_2.

Let us consider a new basic system, namely an elastic body fully clamped on the surface $S = S_1+S_2+S_3$. We assume in this basic system a unitary concentrated mass force acting in the point $(\xi)\in B$ along the x_k-axis. Then the displacements $\overline{G}_i^{(k)}(x,\xi)$, due to this state, should satisfy the following equations of displacement

$$(21)\qquad \mu\,\overline{G}_{i,jj}^{(k)}+(\lambda+\mu)\,\overline{G}_{j,ji}^{(k)}+\delta(x-\xi)\,\delta_{ik}=0\qquad (i,k=1,2,3),$$

with boundary conditions

$$(22)\qquad \overline{G}_i^{(k)}=0\quad \text{on}\quad S.$$

Let us consider the lead and deformation of the elastic body, presented in Fig. 1, as the first system of loads and the displacement $\overline{G}_i^{(k)}$ as being subordinated to the second system of loads. We shall take advantage of Eq. (1). Assuming $v_i = u_i$, $v'_i = \overline{G}_i^{(k)}$, $F_i = X_i$, $F'_i = \delta(x-\xi)\,\delta_{ik}$ and so on, we obtain

$$(23)\qquad \int_B [X_i(x)\,\overline{G}_i^{(k)}(x,\xi)-\delta(x-\xi)\,\delta_{ik}\,u_i(x)]\,dV(x)-\\ -\int_{S_2} u_i(x)\,P_i^{(k)}(x,\xi)\,d\sigma(x)=0\,.$$

We denote here by $P_i^{(k)}(x, \xi)$ the components of the support reaction in the point $(x) \in S_2$, due in the basic system to the action of concentrated unit force applied at the point $(\xi) \in B$ and acting along the x_k-axis

$$P_i^{(k)}(x, \xi) = \bar{\tau}_{ij}^{(k)} n_j = (2\mu \bar{\varepsilon}_{ij}^{(k)} + \lambda \bar{\varepsilon}_{ss}^{(k)} \delta_{ij}) n_j, \tag{24}$$

where

$$\bar{\varepsilon}_{ij}^{(k)} = \frac{1}{2} (\bar{G}_{i,j}^{(k)} + \bar{G}_{j,i}^{(k)}).$$

The relation (23) leads to the equation

$$u_k(\xi) = \bar{u}_k^0(\xi) - \int_{S_2} u_i(x) P_i^{(k)}(x, \xi) \, d\sigma(x), \quad (x) \in S_2, \quad (\xi) \in B. \tag{25}$$

The expression

$$\bar{u}_k^0(\xi) = \int_B X_i(x) \bar{G}_i^{(k)}(x, \xi) \, dV(x)$$

should be considered here as a component of the displacement in the point (ξ) of the basic system, due to the action of mass forces $\bar{X}(x)$. The function $\bar{u}_i^0(x)$ should verify the following equation

$$\mu \bar{u}_{i,jj}^0 + (\lambda + \mu) \bar{u}_{j,ji}^0 + X_i = 0, \tag{26}$$

with boundary conditions

$$\bar{u}_i^0 = 0 \quad \text{on} \quad S. \tag{27}$$

Let us observe that, conformly to J. C. Maxwell's theorem on reciprocity, we have in the basic system

$$P_i^{(k)}(x, \xi) = \bar{U}_k^{(i)}(\xi, x) \quad (\xi) \in B, \quad (x) \in S_2. \tag{28}$$

$\bar{U}_k^{(i)}(\xi, x)$ denotes here a component of the displacement towards x_k in the point (ξ) of the region B, due to the action of concentrated unit displacement acting in the point $(x) \in S_2$ along the x_i-axis.

Eq. (25) may be written in a form more convenient for our further considerations, namely

$$u_k(\xi) = \bar{u}_k^0(\xi) - \int_{S_2} u_i(x) \bar{U}_k^{(i)}(\xi, x) \, d\sigma(x), \quad (k = 1, 2, 3) \quad (\xi) \in B, \quad (x) \in S_2. \tag{29}$$

The unknown functions $u_i(x)$ on the surface S_2 will be determined from the boundary condition on S_2 stating that this surface is under the action of the load $\vec{q}$. Consequently, on the surface S_2 we have

$$g_k(x') = \sigma_{kj}(x') n_j \quad (k = 1, 2, 3) \quad (x') \in S_2. \tag{30}$$

First, we shall determine, using Eq. (29), the stress in the point $(\xi) \in B$. We obtain

$$\sigma_{kj}(\xi) = \bar{\sigma}_{kj}^0(\xi) - \int_{S_2} u_i(x) \, \bar{\tau}_{kj}^{(i)}(\xi, x) \, d\sigma(x). \tag{31}$$

The following notations were here introduced

$$
(32) \qquad \begin{cases} \sigma_{kj} = 2\mu\, \varepsilon_{kj} + \delta_{kj}\, \lambda \varepsilon_{ss}, & \varepsilon_{kj} = \frac{1}{2}(u_{k,j} + u_{j,k}), \\ \bar{\sigma}^0_{kj} = 2\mu\, \bar{\varepsilon}^0_{kj} + \delta_{kj}\, \lambda \bar{\varepsilon}^0_{ss}, & \bar{\varepsilon}^0_{jk} = \frac{1}{2}(\bar{u}^0_{k,j} + \bar{u}^0_{j,k}), \\ \bar{\tau}^{(i)}_{kj} = 2\mu\, \bar{\varepsilon}^{(i)}_{kj} + \delta_{kj}\, \lambda \bar{\varepsilon}^{(i)}_{ss}, & \bar{\varepsilon}^{(i)}_{kj} = \frac{1}{2}(\bar{U}^{(i)}_{k,j} + \bar{U}^{(i)}_{j,k}). \end{cases}
$$

Taking advantage of the condition (30) and Eq. (31) and moving with the point $(\xi) \in B$ to the point $(x') \in S_2$, we get the following system of integral equations

$$
(33) \qquad g_k(x') = n_j\, \bar{\sigma}^0_{kj}(x') - \int_{S_2} u_i(x)\, n_j\, \bar{\tau}^{(i)}_{kj}(x', x)\, d\sigma(x); \qquad (k = 1, 2, 3).
$$

The functions $u_i(x)$, $(x) \in S_2$ being determined from Eq. (33), we are able to determine the displacements $u_k(\xi)$, $k = 1, 2, 3$, in the region B using relation (29).

Finally, let us consider the system presented in Fig. 4 — an elastic body fully clamped on the surfaces S_1 and S_3 and free in the remaining region. We assume as unknown functions of the problem the displacements $u_i(x_\alpha)$, $(x_\alpha) \in S_2'$ on the S' in the section $\alpha - \alpha$. We assume as basic system the elastic body I fully clamped on S_1 and S_α' and the body II fully clamped on the surfaces S_3 and S_α', both being free on the remaining surfaces.

For the region I we get

$$
(34) \qquad u^{\mathrm{I}}_k(\xi_{\mathrm{I}}) = \bar{u}^{\mathrm{I},0}_k(\xi) + \int_{S_\alpha'} u_i(x_\alpha)\, \bar{U}^{\mathrm{I}(i)}_k(\xi_{\mathrm{I}}, x_\alpha)\, d\sigma(x_\alpha), \qquad (x_\alpha) \in S_\alpha', \qquad (\xi_{\mathrm{I}}) \in B_{\mathrm{I}}.
$$

Here, $\bar{u}^{\mathrm{I},0}_k(\xi_{\mathrm{I}})$ stands for the displacement in the point (ξ_{I}) of the basic system I, due to the action of the load $\vec{q}$ and mass forces $\vec{X}$. Then $\bar{U}^{\mathrm{I}(i)}_k(\xi_{\mathrm{I}}, x_\alpha)$ denotes the displacement of the point (ξ_{I}) of the basic system I along the x_k-axis, due to the action of concentrated unit displacement in the point $(x_\alpha) \in S_\alpha'$ acting along the x_i-axis.

A similar equation may be constructed for the point (ξ_{II}) of the system II.

We may obtain the system of three integral equations containing unknown functions $u_i(x_\alpha)$ from the condition stating that in the point $(x_\alpha') \in S_\alpha'$ the forces of mutual interaction of parts I and II are equal — as regards their absolute magnitudes — but acting in opposite directions.

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В. НОВАЦКИЙ, **ФОРМУЛИРОВКА КРАЕВОЙ ЗАДАЧИ ТЕОРИИ УПРУГОСТИ СО СМЕШАННЫМИ КРАЕВЫМИ УСЛОВИЯМИ**

В одной из предыдущих работ [1] автором был разработан метод решения краевых проблем эластостатики при применении способов аналогичных тем, которые применяются в строительной механике, а именно метод сил и деформаций.

В настоящей работе разработаны аналогичные методы решения, опираясь на теорему Бетти и используя функции Грина для более простой системы.

Предложенный метод приводит к решению системы интегральных уравнений Фредгольма первого рода.