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The Plane Non-coupled Dynamic Problem of Thermoelasticity

by

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1. General equations

In Ref. [1] it has been shown that the plane non-coupled dynamic problem of thermoelasticity (for plane strain) can be treated in stresses by means of the stress function F connected with the stresses by the relations

$$\sigma_{ij} = -F_{,ij} + \delta_{ij}\left(F_{,kk} - \frac{1}{2c_2^2}\ddot{F}\right), \qquad i, j = 1, 2, \tag{1.1}$$

and satisfying the non-homogeneous bi-wave equation

$$\Box_1^2 \Box_2^2 F + 2\mu m \Box_2^2 T = 0, \tag{1.2}$$

where T is the temperature, $\mu\lambda$ — the Lamé constants, α_t — the coefficient of thermal dilatation and $m = \frac{(3\lambda+2\mu)\,\alpha_t}{\lambda+2\mu}$, $\Box_1^2, \Box_2^2$ are the differential operators

$$\Box_1^2 = \nabla^2 - \frac{1}{c_1^2}\partial_t^2, \qquad \Box_2^2 = \nabla^2 - \frac{1}{c_2^2}\partial_t^2, \tag{1.3}$$

where ∇^2 is the Laplacian operator and c_1 and c_2 are the velocities of propagation of the longitudinal and transversal elastic waves, respectively,

$$c_1^2 = \frac{\lambda+2\mu}{\varrho}, \qquad c_2 = \frac{\mu}{\varrho}.$$

The other symbols are δ_{ij} — Kronnecker's delta, ϱ — density and ∂_t — the symbol of time derivative.

The temperature T should satisfy the heat equation

$$\Box_3^2 T = -\frac{Q}{\varkappa}, \qquad \Box_3^2 = \nabla^2 - \frac{1}{\varkappa}\partial_t, \tag{1.4}$$

where $\varkappa = \frac{\lambda'}{c\varrho}$, $Q = \frac{W}{c\varrho}$ and λ' is the coefficient of heat conduction, W — the quantity of heat produced per unit volume and time and c — the specific heat.

The solution of the Eq. (1.1) may be expressed in the form of a sum of three functions $F = F_0 + F^*$, $F^* = F_1 + F_2$, where F_0 is a particular integral of (1.2), and F_1, F_2 are the general integrals satisfying the equation

$$\Box_1^2 F_1 = 0, \quad \Box_2^2 F_2 = 0. \tag{1.5}$$

In the case of the infinite space it suffices to determine a particular integral of the equation

$$\Box_1^2 F_0 + 2\mu m T = 0. \tag{1.6}$$

For a limited body, in addition to the particular integral F_0 the functions F_1 and F_2 should be determined. The particular integral F_0 should be selected in such a way that some of the conditions of the problem will be satisfied. The determination of the particular integral may done in the following way.

Let us eliminate the temperature from the equations (1.4) and (1.6). Then,

$$\Box_1^2 \Box_3^2 F_0 = \frac{2\mu m}{\varkappa} Q. \tag{1.7}$$

Bearing in mind that

$$\Box_1^2 \Box_3^2 = \frac{\Box_1^2 - \Box_3^2}{(\Box_3^2)^{-1} - (\Box_1^2)^{-1}}, \quad \Box_1^2 - \Box_3^2 = \frac{1}{\varkappa}\partial_t - \frac{1}{c_1^2}\partial_t^2, \tag{1.8}$$

Eq. (1.7) may be represented in the form:

$$\left(\frac{1}{\varkappa}\partial_t - \frac{1}{c_1^2}\partial_t^2\right) F_0 = -\frac{2\mu m}{\varkappa} [(\Box_3^2)^{-1} - (\Box_1^2)^{-1}] Q. \tag{1.9}$$

Let us introduce an auxiliary function S, satisfying the equation

$$\Box_1^2 S = -Q/\varkappa, \tag{1.10}$$

and assume that function S satisfies the same boundary conditions as function T. Since

$$T = -\frac{1}{\varkappa}(\Box_3^2)^{-1} Q, \quad S = -\frac{1}{\varkappa}(\Box_1^2)^{-1} Q, \tag{1.11}$$

Eq. (1.9) takes the form

$$\left(\frac{1}{\varkappa}\partial_t - \frac{1}{c_1^2}\partial_t^2\right) F_0 = 2\mu m (T - S). \tag{1.12}$$

A number of particular solutions can be obtained in a simple way from (1.12).

Let us assume that the action of the heat sources started at $t > 0$ and for $t \leqslant 0$ the body was in the natural, undeformed and unstressed state. On performing the Laplace transformation on Eq. (1.12), we obtain

$$\widetilde{F}_0 = \frac{2\mu m}{(p/\varkappa - p^2/c_1^2)} (\widetilde{T} - \widetilde{S}), \tag{1.13}$$

where

$$\widetilde{F}_0(x_r, p) = \int_0^\infty (x_r, t)\, e^{-pt}\, dt, \quad \text{etc.}$$

The inverse Laplace transformation yields the function F_0, and the Eqs. (1.1) — the stresses corresponding to this function.

If the heat sources are harmonic in time $Q(x_r, t) = e^{i\omega t} Q_0(x_r)$ the functions T, F_0, S are also harmonic.

Inserting

$$T(x_r, t) = e^{i\omega t} U(x_r), \quad S(x_r, t) = e^{i\omega t} V(x_r), \quad F_0(x_r, t) = e^{i\omega t} \Phi_0(x_r),$$

into (1.12), Φ_0 is obtained from the equation

$$\Phi_0 = \frac{2\mu m (U - V)}{i\eta + k^2}, \quad \eta = \omega/\varkappa, \quad k^2 = \omega^2/c_1^2. \tag{1.14}$$

The stresses corresponding to the function F_0 are expressed by

$$\sigma_{ij} = e^{i\omega t}\left[\Phi_{0,\,ij} + \delta_{ij}\left(\nabla^2 + \frac{\omega^2}{2c_2^2}\right)\Phi_0\right], \quad i, j = 1, 2. \tag{1.15}$$

If the temperature field moves along the x_1-axis with constant velocity v, then, transforming the co-ordinates

$$\xi_1 = x_1 - vt, \quad \xi_2 = x_2, \tag{1.16}$$

Eq. (1.12) is reduced to

$$\left(\frac{v}{\varkappa}\partial_1 + \frac{v^2}{c_1^2}\partial_1^2\right) F_0(\xi_1, \xi_2) = -2\mu m\,[T(\xi_1, \xi_2) - S(\xi_1, \xi_2)], \tag{1.17}$$

where the functions T and S should satisfy the equation

$$\begin{cases} (\partial_1^2 + \partial_2^2 + v/\varkappa\,\partial_1)\,T = -Q/\varkappa, \quad \left(\partial_1^2 + \partial_2^2 - \dfrac{v^2}{c_1^2}\partial_1^2\right) S = -Q/\varkappa, \\ \partial_1 = \dfrac{\partial}{\partial \xi_1}, \quad \partial_2 = \dfrac{\partial}{\partial \xi_2}. \end{cases} \tag{1.18}$$

Let us observe that for an infinite elastic space Eq. (1.13) may also be obtained in another way.

Let us perform the Laplace transformation on Eqs. (1.7) and (1.4). We obtain the system of equations

$$(\nabla^2 - p/\varkappa)(\nabla^2 - p^2/c_1^2)\,\tilde{F}_0 = -\frac{2\mu m}{\varkappa}\tilde{Q}, \tag{1.19}$$

$$(\nabla^2 - p/\varkappa)\,\tilde{T} = -\frac{\tilde{Q}}{\varkappa}. \tag{1.20}$$

Let us perform on (1.19) the triple Fourier integral transformation determined by the equations

$$\begin{cases} f^*(\alpha_r, p) = (2\pi)^{-3/2} \displaystyle\iiint_{-\infty}^{\infty} \tilde{f}(x_r, p)\, e^{i\alpha_r x_r}\, dV, \qquad dV = dx_1\, dx_2\, dx_3, \\ \tilde{f}(x_r, p) = (2\pi)^{-3/2} \displaystyle\iiint_{-\infty}^{\infty} f^*(\alpha_r, p)\, e^{-i\alpha_r x_r}\, dW, \quad dW = d\alpha_1\, d\alpha_2\, d\alpha_3. \end{cases} \tag{1.21}$$

The solution of (1.19) takes the form (2)

$$(1.22)\qquad \widetilde{F}_0(x_r,p) = -\frac{2(2\pi)^{-3/2}\mu m}{\varkappa}\iiint\limits_{-\infty}^{\infty}\frac{Q^*(a_r,p)\,e^{-ia_rx_r}\,dW}{(a_r a_r+p/\varkappa)(a_r a_r+p^2/c_1^2)}$$

or

$$(1.23)\qquad \widetilde{F}_0(x_r,p) = \frac{2(2\pi)^{-3/2}\mu m}{(p/\varkappa-p^2/c_1^2)}\left\{\frac{1}{\varkappa}\iiint\limits_{-\infty}^{\infty}\frac{Q^*\,e^{-ia_rx_r}\,dW}{a_r a_r+p/\varkappa} - \frac{1}{\varkappa}\iiint\limits_{-\infty}^{\infty}\frac{Q^*\,e^{-ia_rx_r}\,dW}{a_r a_r+p^2/c_1^2}\right\}.$$

The first integral in figured brackets on the right-hand side of (1.23) is equal to $\widetilde{T}$, as a solution of (1.20), the second may be treated as a solution of the differential equation

$$(1.24)\qquad (\nabla^2-p^2/c_1^2)\,\widetilde{S} = -\frac{Q}{\varkappa},\qquad S(x_r,0)=\dot{S}(x_r,0)=0.$$

It should be added that for the determination of the function $\widetilde{F_0}$ it suffices to know the transforms of the function $\widetilde{T}$. The function $\widetilde{S}$ will be obtained by replacing $p/\varkappa$ in the transform $\widetilde{T}$ by p^2/c_1^2.

Below a few solution examples are given for an infinite elastic space and an elastic layer.

2. The infinite space

Let an instantaneous linear source uniformly distributed along the x_3-axis act in the infinite space $Q(x_r,t) = Q_0\dfrac{\delta(r)}{2\pi r}\delta(t)$. The resulting temperature field is

$$(2.1)\qquad T(r,t) = \frac{Q_0}{4\pi\varkappa t}\exp\left(-\frac{r^2}{4\varkappa t}\right),\qquad r^2 = x_1^2+x_2^2.$$

On performing the Laplace transformation on Eq. (2.1), we obtain

$$(2.2)\qquad \widetilde{T}(r,p) = \frac{Q_0}{2\pi\varkappa}K_0(r\sqrt{p/\varkappa}),$$

where $K_0(z)$ is the modified Bessel function of the third kind and of zero order.

Replacing $p/\varkappa$ in (2.2) by p^2/c_1^2, we obtain

$$(2.3)\qquad \widetilde{S}(r,p) = \frac{Q_0}{2\pi\varkappa}K_0(rp\sigma),\qquad \sigma = 1/c_1.$$

Therefore

$$(2.4)\qquad \widetilde{F}_0(r,p) = \frac{Q_0\mu m}{2\pi\varkappa(p/\varkappa-p^2/c_1^2)}\left[K_0(r\sqrt{p/\varkappa})-K_0(rp\sigma)\right].$$

On performing the inverse Laplace transformation, we obtain the function F_0. The same problem has been solved in another way by W. Derski [3].

In the case of a periodic source $Q(x_r, t) = Q_0 e^{i\omega t} \frac{\delta(r)}{2\pi r}$, we obtain from (1.14)

$$\Phi_0 = \frac{\mu m Q^0}{i\eta + k^2} [K_0(r\sqrt{i\eta}) - K_0(rk)], \quad \eta = \omega/\varkappa, \quad k = \omega/c_1. \tag{2.5}$$

The stresses σ_{ij}, corresponding to the function Φ_0, will be obtained from Eqs. (1.15)

Let the heat sources vary harmonically in time along the x_2-axis

$$Q(x_1, x_2, t) = Q_0 e^{i\omega t + i\alpha_n x_2} \delta(x_1), \quad \alpha_n = \frac{n\pi}{a}, \quad n = 0, \pm 1, \pm 2, \ldots .$$

In this case the following equation is to be solved

$$\nabla^2 U - i\eta U = -\frac{Q_0}{\varkappa} \delta(x_1) e^{i\alpha_n x_2}. \tag{2.6}$$

It is easy to verify that the following function is a solution of this equation

$$U = \frac{Q_0}{2\varkappa} e^{i\alpha_n x_2} \frac{e^{-\vartheta_n x_1}}{\vartheta_n}, \quad \vartheta_n = \sqrt{\alpha_n^2 + i\eta}, \quad x_1 > 0. \tag{2.7}$$

Therefore

$$V = \frac{Q_0}{2\varkappa} e^{i\alpha_n x_2} \frac{e^{-\gamma_n x_1}}{\gamma_n}, \quad \gamma_n = \sqrt{\alpha_n^2 - k^2}, \quad k = \omega/c_1, x_1 > 0. \tag{2.8}$$

Substituting (2.7) and (2.8) into Eq. (1.14), we find

$$\Phi_0 = \frac{Q_0 \mu m e^{i\alpha_n x_2}}{\varkappa(i\eta + k^2)} \left(\frac{e^{-\vartheta_n x_1}}{\vartheta_n} - \frac{e^{-\gamma_n x_1}}{\gamma_n} \right), \quad x_1 > 0. \tag{2.9}$$

Knowing Φ_0, the stresses will be found from Eqs. (1.15).

3. The elastic layer

Let us consider an elastic layer of thickness $2h$ with boundary surfaces free from stress. The same surfaces are heated harmonically in time according to the equation

$$T(\pm h, x_2, t) = e^{i\omega t} \sum_{n=-\infty}^{n=\infty} T_n e^{i\alpha_n x_2} = \tag{3.1}$$

$$= e^{i\omega t} \left[\frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n \cos \alpha_n x_2 + B_n \sin \alpha_n x_2) \right], \quad \alpha_n = n\pi/a.$$

In order to determine the particular integral F_0 the following equation should be solved

$$\nabla^2 U - i\eta U = 0, \tag{3.2}$$

with the boundary conditions

$$U(h, x_2) = \sum_{n=-\infty}^{n=\infty} T_n e^{i\alpha_n x_2} \quad \text{and} \quad U = 0 \quad \text{for} \quad x_1^2 + x_2^2 \to \infty . \tag{3.3}$$

The solution of Eq. (3.2) with the boundary conditions (3.3) has the form

$$U(x_1, x_2) = \sum_{n=-\infty}^{n=\infty} T_n \frac{\operatorname{ch} \vartheta_n x_1}{\operatorname{ch} \vartheta_n h} e^{i\alpha_n x_2}, \quad \vartheta_n = \sqrt{\alpha_n^2 + i\eta}. \tag{3.4}$$

Therefore

$$V(x_1, x_2) = \sum_{n=-\infty}^{n=\infty} T_n \frac{\operatorname{ch} \gamma_n x_1}{\operatorname{ch} \gamma_n h} e^{i\alpha_n x_2} \quad \gamma_n = \sqrt{\alpha_n^2 - k^2}, \tag{3.5}$$

and

$$\Phi_0 = \frac{2\mu m}{i\eta + k^2} \sum_{n=-\infty}^{n=\infty} T_n e^{i\alpha_n x_2} \left(\frac{\operatorname{ch} \vartheta_n x_1}{\operatorname{ch} \vartheta_n h} - \frac{\operatorname{ch} \gamma_n x_1}{\operatorname{ch} \gamma_n h} \right). \tag{3.6}$$

Thus, a particul ar integral of Eq. (1.2) has been obtained, satisfying a part of the boundary c onditions, the normal stresses σ_{11} on the boundaries $x_1 = \pm h$ being zero. The shear stresses σ_{12} remain different from zero. To satisfy all the boundary conditions, particular integrals of (1.2) should be taken into consideration. Introducing the equation

$$F_1(x_r, t) = e^{i\omega t} \Phi_1(x_r), \quad F_2(x_r, t) = e^{i\omega t} \Phi_2(x_r), \tag{3.7}$$

Eq. (1.5) is reduced to the form

$$(\nabla^2 + k^2) \Phi_1 = 0, \quad (\nabla^2 + s^2) \Phi_2 = 0, \quad k^2 = \frac{\omega^2}{c_1^2}, \quad s^2 = \frac{\omega^2}{c_2^2}. \tag{3.8}$$

The solutions of these equations will be assumed in the form

$$\begin{cases} \Phi_1 = \sum_{n=-\infty}^{n=\infty} Q_n e^{i\alpha_n x_2} \operatorname{ch} \gamma_n x_1, \\ \Phi_2 = \sum_{n=-\infty}^{n=\infty} b_n e^{i\alpha_n x_2} \operatorname{ch} \beta_n x_1, \quad \beta_n = \sqrt{\alpha_n^2 - s^2}. \end{cases} \tag{3.9}$$

From the boundary conditions $\sigma_{12} = 0$, $\sigma_{11} = 0$ for $x_1 = h$, that is from the conditions

$$-\Phi_{,12} = 0, \quad \Phi_{,22} + \frac{s^2}{2} \Phi = 0, \quad \Phi = \Phi_0 + \Phi_1 + \Phi_2, \tag{3.10}$$

the constants a_n and b_n can be determined. We find

$$\begin{cases} Q_n = \dfrac{2\mu m T_n}{i\eta + \sigma^2} \operatorname{ch} \beta_n h \dfrac{\vartheta_n \operatorname{th} \vartheta_n h - \gamma_n \operatorname{th} \gamma_n h}{\gamma_n \operatorname{sh} \gamma_n h \operatorname{ch} \beta_n h - \beta_n \operatorname{sh} \beta_n h \operatorname{ch} \gamma_n h}, \\ b_n = -a_n \dfrac{\operatorname{ch} \gamma_n h}{\operatorname{ch} \beta_n h}. \end{cases} \tag{3.11}$$

The knowledge of the function $\Phi = \Phi_0 + \Phi_1 + \Phi_2$ will enable us to determine the stresses from the equations

$$\sigma_{ij} = e^{i\omega t}\left[-\Phi_{,ij} + \delta_{ij}\left(\nabla^2 + \frac{s^2}{2}\right)\Phi\right], \quad i,j = 1,2. \tag{3.12}$$

Considerable simplification is obtained in the case of the boundary condition

$$T(\pm h, x_2, t) = T_0 e^{i\omega t}, \tag{3.13}$$

where T does not depend on x_2. In this case the solution of the problem is confined to the function

$$F_0 = \frac{2\mu m T_0 e^{i\omega t}}{i\eta + k^2}\left(\frac{\operatorname{ch} x_1 \sqrt{i\eta}}{\operatorname{ch}\sqrt{i\eta}\, h} - \frac{\cos kx_1}{\cos kh}\right). \tag{3.14}$$

Making use of Eqs. (1.15) we find that the boundary conditions $\sigma_{12} = 0$, $\sigma_{11} = 0$ are satisfied here. The stresses in the elastic layer are

$$\sigma_{11} = -\frac{1}{2c_2^2}\ddot{F}_0, \quad \sigma_{22} = F_{0,11} - \frac{1}{2c_2^2}\ddot{F}_0, \quad \sigma_{12} = 0. \tag{3.15}$$

Thus, we have forced vibration produced by a harmonic temperature field. The excitation frequency ω should not coincide with the natural vibration frequency of the layer $\omega_n = \frac{2n-1}{2}\frac{\pi c_1}{h}$ $n = 1, 2, \ldots, \infty$. Otherwise we would have the phenomenon of resonance. The quantities ω_n are the roots of the equation $\cos kh = 0$. For ω approaching anyone of the ω_n values, the stresses will increase indefinitely.

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