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Application of Difference Equations in Structural Mechanics. I

by

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1. Introduction

Many problems of structural mechanics lead to linear differential equations either ordinary or partial with variable coefficients. In view of mathematical difficulties which we meet in solving exactly the equations, we replace the differential equations by difference equations, substituting difference quotients for derivatives.

In many problems of statics, stability and dynamics of plates and shells the highest order derivatives have constant coefficients, while the derivatives of lower orders appear with variable coefficients. As an example we may quote the equation of deflection of a plate simultaneously bent and compressed

$$N\nabla^4 w + q(x,y)\frac{\partial^2 w}{\partial x^2} + r(x,y)\frac{\partial^2 w}{\partial y^2} + s(x,y)\frac{\partial^2 w}{\partial x\,\partial y} = p(x,y). \tag{1.1}$$

Here $w(x,y)$ is the plate deflection, $N = \text{const}$ — the plate rigidity, $p(x,y)$ — the loading of the plate normal to the plate surface, and $q(x,y)$, $r(x,y)$, $s(x,y)$ are loadings acting in the plate surface.

Eq. (1.1) can be represented in the operator form

$$\mathcal{L}(w) = p - \mathcal{D}(w), \tag{1.2}$$

where $\mathcal{L}(w) = N\nabla^4 w$ and $\mathcal{D}(w)$ are linear differential operators.

The derivatives in the $\mathcal{L}(w)$ operator are multiplied by constant coefficients, while derivatives in the operator $\mathcal{D}(w)$ — by variable coefficients. The solution of Eq. (1.2) can be reduced to the integro-differential equation

$$w = \int_\sigma p\overline{w}\,d\sigma - \int_\sigma \overline{w}\,\mathcal{D}(w)\,d\sigma, \tag{1.3}$$

or, for some definite boundary conditions, to an integral equation [1], [2]. $\overline{w}(x, y; \xi, \eta)$ denotes Eq. (1.3) the Green function satisfying the equation

$$\mathcal{L}(w) = \delta(x-\xi)\,\delta(y-\eta), \tag{1.4}$$

where δ is the Dirac delta.

Difference equations can be solved in the same way. If we substitute in Eq. (1.1) difference quotients for derivatives, then we can reduce this equation to the form

$$\mathcal{L}_{xy}(w_{xy}) = p_{xy} - \mathcal{D}_{xy}(w_{xy}), \tag{1.5}$$

where

$$\mathcal{L}_{xy}(w_{xy}) = N(\Delta_x^4 + 2\varepsilon^2 \Delta_x^2 \Delta_y^2 + \varepsilon^4 \Delta_y^4)\, w_{xy}, \tag{1.6}$$

$$\mathcal{D}_{xy}(w_{xy}) = (\bar{q}_{xy} \Delta_x^2 + \bar{r}_{xy}\, \varepsilon^2 \Delta_y^2 + \bar{s}_{xy}\, \varepsilon \Delta_x \Delta_y)\, w_{xy}, \tag{1.7}$$

$\bar{q}_{xy} = q_{xy}\, \Delta x^2$, and so on, where $\varepsilon = \Delta x / \Delta y$.

The difference equations of the type (1.5) are to be solved in the statics of frame systems, in flat gridworks, and for beams on elastic supports. For this reason the way of solution of these equations can be widely applied in practice.

It is essential for further considerations that the operator $\mathcal{L}_{xy}$ contains the difference quotients of the highest orders with respect to the directions x and y, respectively.

2. Solution of the difference equation $\mathcal{L}_{xy}(w_{xy}) = p_{xy} - \mathcal{D}_{xy}(w_{xy})$

We shall seek the solution of Eq. (1.5) in the form of the sum equation

$$w_{xy} = \sum_{\xi, \eta}^{n, m} q_{\xi\eta}\, \bar{w}_{\xi\eta xy} - \sum_{\xi, \eta}^{n, m} \mathcal{D}_{\xi\eta}(w_{\xi\eta})\, \bar{w}_{\xi\eta xy}. \tag{2.1}$$

Here $\bar{w}_{xy\xi\eta}$ is the solution of the difference equation

$$\mathcal{L}_{xy}(\bar{w}_{xy\xi\eta}) = \delta_{x\xi}\, \delta_{y\eta}, \tag{2.2}$$

where

$$\delta_{x\xi} = \begin{cases} 1 \text{ for } x = \xi, \\ 0 \text{ for } x \neq \xi, \end{cases} \qquad \delta_{y\eta} = \begin{cases} 1 \text{ for } y = \eta, \\ 0 \text{ for } y \neq \eta. \end{cases} \tag{2.3}$$

Function $\bar{w}_{xy\xi\eta}$ is the Green function (the influence surface) of the equation $\mathcal{L}_{xy}(w_{xy}) = f_{xy}$ and the functions $\delta_{x\xi}$ and $\delta_{y\eta}$ in the difference equation are the Dirac deltas. It is easy to prove that they satisfy the following relations:

$$\sum_{x=1}^{n} \delta_{x\xi}\, k_{xy} = k_{\xi y}, \qquad \sum_{y=1}^{n} \delta_{y\eta}\, g_{xy} = g_{x\eta}, \qquad \sum_{x, y}^{n, m} \delta_{x\xi}\, \delta_{y\eta}\, h_{xy} = h_{\xi\eta}, \tag{2.4}$$

and

$$\sum_{x=1}^{n} \delta_{x\xi} = 1, \qquad \sum_{y=1}^{n} \delta_{y\eta} = 1, \qquad \sum_{x, y}^{n, m} \delta_{x\xi}\, \delta_{y\eta} = 1. \tag{2.5}$$

In order to determine the Green functions, the knowledge of the eigenfunction φ_{xy}^{ik}, satisfying the equation, written below, is necessary [3]:

$$\mathcal{L}_{xy}(\varphi_{xy}^{ik}) = \sigma_{ik}\, \varphi_{xy}^{ik}, \qquad i = 1, 2, ..., n; \qquad k = 1, 2, ..., m. \tag{2.6}$$

The functions φ_{xy}^{ik} should satisfy the same boundary conditions as the function w_{xy}. We assume that the functions φ_{xy}^{ik} and the coefficients σ_{ik} are known, and that they form an orthonormal system, i.e.

$$\sum_{x, y}^{n, m} \varphi_{xy}^{ik}\, \varphi_{xy}^{\nu\mu} = \delta_{i\nu}\, \delta_{k\mu}. \tag{2.7}$$

Let us expand the functions $\bar{w}_{xy\xi\eta}$ and $\delta_{x\xi}, \delta_{y\eta}$ into double series of eigenfunctions

$$\bar{w}_{xy\xi\eta} = \sum_{i,k} c_{ik}\,\varphi^{ik}_{xy}, \qquad \delta_{x\xi}\,\delta_{y\eta} = \sum_{i,k} a_{ik}\,\varphi^{ik}_{xy}, \tag{2.8}$$

where, taking into account (2.4), we have

$$a_{ik} = \sum_{x,y} \delta_{x\xi}\,\delta_{y\eta}\,\varphi^{ik}_{xy} = \varphi^{ik}_{\xi\eta}. \tag{2.9}$$

Substituting (2.8) into Eq. (2.2), we obtain

$$\sum_{i,k} c_{ik}\,\mathcal{L}_{xy}\left(\varphi^{ik}_{xy}\right) = \sum_{i,k} \varphi^{ik}_{\xi\eta}\,\varphi^{ik}_{xy}. \tag{2.10}$$

Taking into account Eq. (2.6), we obtain from (2.10); $\sigma_{ik}\,c_{ik} = \varphi^{ik}_{\xi\eta}$.
Thus,

$$\bar{w}_{xy\xi\eta} = \sum_{i,k} \frac{1}{\sigma_{ik}}\,\varphi^{ik}_{xy}\,\varphi^{ik}_{\xi\eta}. \tag{2.11}$$

Now we substitute Eq. (2.11) into Eq. (2.1) and expand the function $q_{\xi\eta}$ into the series of functions $\varphi^{\nu\mu}_{\xi\eta}$

$$\begin{aligned} w_{xy} = \sum_{\xi,\eta} \sum_{\nu,\mu} q_{\nu\mu}\,\varphi^{\nu\mu}_{\xi\eta} \sum_{i,k} \frac{1}{\sigma_{ik}}\,\varphi^{ik}_{xy}\,\varphi^{ik}_{\xi\eta} - \\ - \sum_{\xi,\eta} \mathcal{D}_{\xi\eta}\left(w_{\xi\eta}\right) \sum_{i,k} \frac{1}{\sigma_{ik}}\,\varphi^{ik}_{xy}\,\varphi^{ik}_{\xi\eta}, \qquad q_{\nu\mu} = \sum_{\nu,\mu} q_{\xi\eta}\,\varphi^{\nu\mu}_{\xi\eta}. \end{aligned} \tag{2.12}$$

Finally, expressing the function w_{xy} by means of the series

$$w_{xy} = \sum_{\nu,\mu} A_{\nu\mu}\,\varphi^{\nu\mu}_{xy}, \tag{2.13}$$

and inserting it into Eq. (2.12), after simple transformations and change of summation, we obtain, taking into account relations (2.4), (2.5) and (2.7), a system of non-homogeneous linear equations:

$$A_{\nu\mu}\,\sigma_{\nu\mu} = q_{\nu\mu} - \sum_{i,k} A_{ik}\,b_{ik\nu\mu}, \qquad i, \nu = 1, 2, \ldots, n, \qquad k, \mu = 1, 2, \ldots, m, \tag{2.14}$$

where

$$b_{ik\nu\mu} = \sum_{\xi,\eta} \varphi^{\mu\nu}_{\xi\eta}\,\mathcal{D}_{\xi\eta}\left(\varphi^{ik}_{\xi\eta}\right).$$

After the determination of quantities $A_{\nu\mu}$ we get function w_{xy} from relation (2.13). It is interesting that we perform eventually the operation $\mathcal{D}_{\xi\eta}$ on the known function $\varphi^{ik}_{\xi\eta}$.

Although for the determination of the quantities $A_{\nu\mu}$ we have $n \cdot m$ equations, we can confine ourselves for calculations to the system of several equations with the same number of unknowns, because the successive terms on the right-hand side of Eq. (2.14) are rapidly decreasing. In the case $q_{\nu\mu} = 0$ we have a system of homogeneous equations. The parameters appearing there (as for example the frequency of free vibrations, the critical force) can be determined from the secular equation having the form

$$\|\sigma_{\nu\mu}\,\delta_{ik}\,\delta_{\mu\nu}+b_{ik\nu\mu}\|=0. \tag{2.15}$$

We obtain a particularly simple solution of the system of equations when the operator $\mathcal{D}_{\xi\eta}$ contains differences with constant coefficients. In this case we have

$$\mathcal{D}_{\xi\eta}(\varphi^{ik}_{\xi\eta})=\Phi_{ik}\,\varphi^{ik}_{\xi\eta}, \tag{2.16}$$

where Φ_{ik} have constant values. Thus

$$b_{ik\nu\mu}=\Phi_{ik}\sum_{\xi,\eta}\varphi^{\nu\mu}_{\xi\eta}\,\varphi^{ik}_{\xi\eta}=\Phi_{ik}\,\delta_{ik}\,\delta_{\nu\mu}, \tag{2.17}$$

and Eq. (2.14) takes the form

$$A_{\nu\mu}(\sigma_{\nu\mu}+\Phi_{\nu\mu})=q_{\nu\mu},\quad \nu=1,2,\ldots,n,\quad \mu=1,2,\ldots,m. \tag{2.18}$$

Substituting (2.18) into Eq. (2.13), we obtain

$$w_{xy}=\sum_{\nu,\mu}\frac{q_{\nu\mu}}{\sigma_{\nu\mu}+\Phi_{\nu\mu}}\varphi^{\nu\mu}_{xy}. \tag{2.19}$$

The proposed way of solving of Eq. (1.5) can obviously be extended to difference equations containing a greater number of variables.

3. Examples

Let us consider a rectangular plate simply supported on the whole boundary and compressed in the x direction by the forces $q(x, y)$. Replacing the differential equation by the difference equation, we obtain

$$\mathcal{L}_{xy}(w_{xy})=-\mathcal{D}_{xy}(w_{xy}), \tag{3.1}$$

where the operator $\mathcal{L}_{xy}$ is given by Eq. (1.7), and

$$\mathcal{D}_{xy}(w_{xy})=\bar{q}_{xy}\,\Delta^2_x(w_{xy}),\qquad \bar{q}_{xy}=q_{xy}\,\Delta x^2. \tag{3.2}$$

The eigenfunctions for the assummed conditions of the simple support are of the form

$$\varphi^{ik}_{\nu y}=X^i_x\,Y^k_y,\qquad X^i_x=\sqrt{\frac{2}{n}}\sin\alpha_i x,\qquad Y^k_y=\sqrt{\frac{2}{m}}\sin\beta_k y,$$
$$\alpha_i=\frac{i\pi}{n},\qquad \beta_k=\frac{k\pi}{m}. \tag{3.3}$$

Substituting (3.3) into Eq. (2.6), we obtain

$$\sigma_{ik}=2N[\cos 2\alpha_i-4\cos\alpha_i+6+4\varepsilon^2(\cos\alpha_i-1)(\cos\beta_k-1)+$$
$$+\varepsilon^4(\cos 2\beta_k-4\cos\beta_k+6)]. \tag{3.4}$$

Now we assume that $\bar{q}_{xy}=\bar{q}\,\delta_{y\bar{y}}$. It means that the loading q_{xy} acts along the line $y=\bar{y}$. From Eq. (2.14) we calculate

$$b_{ik\nu\mu}=\bar{q}\sum_{\xi,\eta}\varphi^{\nu\mu}_{\xi\eta}\,\delta_{\eta\bar{\eta}}\,\Delta^2_\xi(\varphi^{ik}_{\xi\eta})=-2r_i\,\bar{q}\sum_{\xi}X^\nu_\xi\,X^i_\xi\sum_{\eta}\delta_{\eta\bar{\eta}}\,Y^k_\eta\,Y^\mu_\eta=$$
$$=-2r_i\,\bar{q}\,\delta_{i\nu}\,Y^k_{\bar{\eta}}\,Y^\mu_{\bar{\eta}},\qquad r_i=1-\cos\alpha_i. \tag{3.5}$$

Eq. (2.14) takes the form

(3.6)
$$A_{\nu\mu}=\frac{2\bar{q}r_\nu Y^{\mu}_{\eta}}{\sigma_{\nu\mu}}\sum_k A_{\nu k}Y^{k}_{\eta}.$$

Now we multiply both sides of Eq. (3.5) by Y^{μ}_{η} and sum with respect to μ.

$$\sum_\mu A_{\nu\mu}Y^{\mu}_{\eta}=2\bar{q}\sum_\mu\frac{r_\nu (Y^{\mu}_{\eta})^2}{\sigma_{\nu\mu}}\sum_k A_{\nu k}Y^{k}_{\eta}.$$

The sum on the left-hand side of the equation is equal to the second sum on the right-hand side of the equation; thus we have

(3.6)
$$\bar{q}=\frac{1}{2}\cdot\frac{1}{\sum_\mu\frac{r_\nu (Y^{k}_{\eta})^2}{\sigma_{\nu\mu}}}.$$

Depending on the number of half-waves in the direction x, we assume in turn $\nu=1,2,\dots$.

In the particular case of $\bar{q}_{xy}=\text{const}$ in the whole plate region, taking into account that

$$b_{ik\nu\mu}=-\bar{q}r_i\,\delta_{i\nu}\,\delta_{k\mu},$$

we obtain from Eq. (2.14)

(3.7)
$$\sigma_{\nu\mu}=\bar{q}r_\nu.$$

Let us consider a continuous beam resting on elastic supports of rigidity λ_x. Let the rigidity of the beam EI be equal in all spans l. We denote the support moments by M_x, and by S the compressive force constant in all spans. In order to solve the problem of buckling of a beam with variable rigidities λ_x, we obtain the system of the following difference equations (cf. [4]) connecting the moments M_x and the support deflections:

(3.8)
$$\begin{cases} G_x(M_x)+\dfrac{EI}{l^2}\Delta_x^2(\delta_x)=0,\\ \Delta_x^2(M_x)-(S\Delta_x^2+\lambda_x l)\,\delta_x=0.\end{cases}$$

Introducing the notations

$$\frac{EI}{l^2}\delta_x=w_x,\qquad S\delta_x=\gamma^2 w_x,\qquad \gamma^2=\frac{Sl^2}{EI},\qquad \varkappa_x=\frac{\lambda_x l^3}{EI},$$

and eliminating from Eq. (3.8) moments M_x, we obtain the equation

(3.9)
$$\mathcal{L}_x(w_x)=-\mathcal{D}_x(w_x),$$

where

$$\mathcal{L}_x=(\Delta_x^4+\gamma^2 G_x\Delta_x^2)\,w_x,\qquad \mathcal{D}_x(w_x)=\varkappa_x G_x(w_x),$$

$$G_x(w_x)=j(\gamma)\,w_{x-1}+2i(\gamma)\,w_x+j(\gamma)\,w_{x+1},$$

and
$$i(\gamma)=\frac{1-\gamma\operatorname{ctg}\gamma}{\gamma^2},\qquad j(\gamma)=\frac{1}{\gamma^2}\left(\frac{\gamma}{\sin\gamma}-1\right).$$

The solution of Eq. (3.9) is reduced to the determination of coefficients A_ν from the system of equations

$$A_s = -\frac{1}{\sigma_\nu}\sum_s A_s b_{s\nu}, \qquad b_{s\nu} = \sum_\xi \varphi_\xi^\nu \mathcal{D}_\xi(\varphi_\xi^s), \qquad \nu, s = 1, 2, \ldots, n. \tag{3.10}$$

The function w_ν is obtained from the equation

$$w_x = \sum_\nu A_\nu \varphi_x^\nu. \tag{3.11}$$

Assuming that the ends of the beam are simply supported, we have

$$\varphi_\xi^\nu = \sqrt{\frac{2}{n}}\sin\gamma_\nu\xi, \qquad \gamma_\nu = \frac{\nu\pi}{n}. \tag{3.12}$$

Substituting (3.12) into the equation

$$\mathcal{L}_x(\varphi_x^\nu) = \sigma_\nu \varphi_x^\nu, \tag{3.13}$$

we obtain

$$\sigma_\nu = (2\cos 2\gamma_\nu + 8\cos\gamma_\nu + 6) + \gamma^2 [2j_\nu \cos 2\gamma_\nu + 4(i_\nu - j_\nu)\cos\gamma_\nu + 2(j_\nu - 2i_\nu)]. \tag{3.14}$$

Now we assume that the function $\varkappa_x$ characterizing the variation of the supports rigidity varies in the following way

$$\varkappa_x = \varkappa_0(1 + \delta_{x\bar{x}}). \tag{3.15}$$

It means that the rigidity of the support $x = \bar{x}$ is twice as great as the rigidity of others. Thus,

$$b_{s\nu} = \varkappa_0 \vartheta_s(\delta_{s\nu} + \varphi_{\bar{\xi}}^\nu \varphi_{\bar{\xi}}^s), \qquad \vartheta_s = 2(j_s \cos\alpha_s + i_s),$$

and consenquently Eq. (3.10) takes the form

$$A_\nu(\sigma_\nu + \varkappa_0 \vartheta_{s\nu}) = -\varkappa_0 \varphi_{\bar{\xi}}^\nu \sum_s A_s \vartheta_s \varphi_{\bar{\xi}}^s. \tag{3.16}$$

If the rigidities of the supports are identical ($\varkappa_x = \varkappa_0 = \text{const}$), then from the system of Eq. (3.16) we obtain the condition of stability

$$\sigma_\nu + \varkappa_0 \vartheta_\nu = 0, \qquad \nu = 1, 2, \ldots, n. \tag{3.17}$$

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