BULLETIN
DE
L'ACADÉMIE POLONAISE DES SCIENCES

CLASSE QUATRIÈME

VOLUME V
NUMÉRO 4

VARSOVIE 1957

BULLETIN DE L'ACADEMIE
POLONAISE DES SCIENCES
Cl. IV — Vol. V, No. 4, 1957

APPLIED MECHANICS

A Boundary Problem of Heat Conduction

by

W. NOWACKI

Presented on April 4, 1957

Consider an elastic body with a steady concentrated heat source of intensity W located at a point Q inside the body. The action of this heat source causes in the elastic body a temperature field T and a state of stress (σ_{ij}). Assume that $T = 0$ on the surfaces Γ_a, Γ_b, constituting part of the total surface of the body, the boundary condition for the remaining part, Γ_c, being $\partial T/\partial n = 0$. This means that there is no heat flow in the normal direction n, in other words, that the surface Γ_c is thermally insulated (Fig. 1). Our problem is to determine the temperature field in the body considered.

We should therefore solve the heat equation [1]

$$\nabla^2 T(P) = -\frac{W}{\varkappa}\delta(Q), \tag{1}$$

with the boundary conditions

$$\begin{aligned} T(S') &= 0 \quad \text{on} \quad \Gamma_a \text{ and } \Gamma_b, \\ \frac{\partial T(S')}{\partial n} &= 0 \quad \text{on} \quad \Gamma_c, \end{aligned} \tag{2}$$

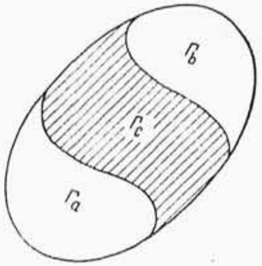

Fig. 1

where P denotes any point inside the body and S'-any point on its surface. We have also $\varkappa = \lambda/\varrho c$, where λ is the coefficient of heat conduction, ϱ density and c specific heat. Finally, δ denotes the Dirac function. Let us observe that the unknown function on the surface Γ_a and Γ_b is the function $\psi(S') = \partial T(S')/\partial n$ of the temperature gradient, and on the surface Γ_c — the function $\varphi(S') = T(S')$.

The solution of the Eq. (1) will be assumed in the integral form

$$T(P) = T_0(P) + \iint_{(\Gamma_a)} \psi(S)\, G(P,S)\, d\Gamma_a, \tag{3}$$

where S denotes any point on the surface Γ_a. In the relation (3), $T_0(P)$ denotes the temperature field satisfying the differential equation

$$\nabla^2 T_0(P) = -\frac{W}{\varkappa}\delta(Q) \tag{4}$$

and the boundary conditions

$$\begin{aligned} T_0(S') &= 0 \quad \text{on} \quad \Gamma_b, \\ \frac{\partial T_0(S')}{\partial n} &= 0 \quad \text{on} \quad \Gamma_a \text{ and } \Gamma_c. \end{aligned} \tag{5}$$

The function $G(P, S)$ is the Green function determining the temperature field satisfying the differential equation

$$\nabla^2 G(P,S) = 0 \tag{6}$$

and the boundary conditions

$$\begin{aligned} G(S,S') &= 0 \quad \text{on} \quad \Gamma_b, \\ \frac{\partial G(S,S')}{\partial n} &= \delta(S'-S) \quad \text{on} \quad \Gamma_a \text{ and } \Gamma_c. \end{aligned} \tag{7}$$

We assume that the point S' lies on the surface Γ_a so that

$$\iint_{(\Gamma_a)} \delta(S'-S)\, d\Gamma_a = 1 \qquad \iint \delta(S'-S)\, d\Gamma_b = 0. \tag{8}$$

The function $\psi(S) = \partial T(S)/\partial n$ appearing on the surface Γ_a is chosen in such a way that the condition $T(S') = 0$ is satisfied on the surface Γ_a.

From the Eq. (3) we obtain

$$T(S') = 0 = T_0(S') + \iint_{(\Gamma_a)} \psi(S)\, G(S',S)\, d\Gamma_a. \tag{9}$$

This is Fredholm's integral equation of the first type. From the solution of this equation we obtain the function $\psi(S)$, the knowledge of which will enable us to determine the temperature field, which is sought, from the Eq. (3). Shifting the point P to S' on the edge of Γ_a and differentiating the function T with respect to the normal, we have from the Eq. (3)

$$\frac{\partial T(S')}{\partial n} = \frac{\partial T_0(S')}{\partial n} + \iint_{(\Gamma_a)} \psi(S)\frac{\partial G(S',S)}{\partial n}\, d\Gamma_a. \tag{10}$$

Bearing in mind the second of the boundary conditions (5) and the second of the boundary conditions (7), we find that

$$\frac{\partial T(S')}{\partial n} = \iint_{(\Gamma_a)} \psi(S)\,\delta(S'-S)\, d\Gamma_a = \psi(S'). \tag{11}$$

It is seen that the integral expression (3), where the function $\psi(S)$ is the result of integration of the Eq. (9), satisfies the differential equation of heat conduction (1) with the boundary conditions (2) assumed above.

Our solution can easily be extended to the case of a heat source distributed along any curve or over any surface or space region inside the body.

In the case, where no heat source appears inside the body and a function $T(S')$ is assumed on the surface Γ_a, the Eq. (3) takes the form

$$T(P) = \iint_{(\Gamma_a)} \psi(S)\, G(P, S)\, d\,\Gamma_a. \tag{12}$$

The function $\psi(S)$ will be obtained by solving the integral equation

$$T(S') = \iint_{(\Gamma_a)} \psi(S)\,(S', S)\, d\,\Gamma_a. \tag{13}$$

This solution method may be extended to cases, where different boundary conditions appear on more than three parts of the surface bounding the elastic body and, finally, to cases of thermal insulation on the surfaces $\Gamma_i\ (i = 1, 2, \dots r)$ inside the body.

The procedure of the above method will be illustrated by a few simple examples:

(a) Consider a concentrated heat source of intensity W at a point Q of co-ordinates $(\bar{\xi}, \bar{\eta}, \bar{\zeta})$ inside the body represented in Fig. 2. Let $T = 0$ on the planes $x = 0$, a and $y = 0$, b. Further, on the plane Γ_a, let $T = 0$, and, on the plane Γ_c, let $\partial T/\partial z = 0$.

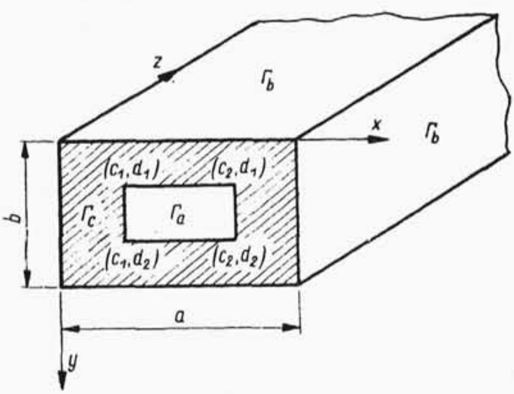

Fig. 2

The temperature field will be found from the Eq. (3) which, in our case, has the form

$$T(x, y, z) = T_0(x, y, z) + \int_{c_1}^{c_2}\int_{d_1}^{d_2} \psi(\xi, \eta)\, G(x, y, z; \xi, \eta, 0)\, d\xi\, d\eta. \tag{14}$$

The function $T_0(x, y, z)$, will be obtained by solving the equation

$$\nabla^2 T_0 = -\frac{W}{\varkappa}\, \delta(x - \bar{\xi})\, \delta(y - \bar{\eta})\, \delta(z - \bar{\zeta}) \tag{15}$$

with the boundary conditions

$$T = 0 \quad \text{for} \quad x = 0, a \quad \text{and} \quad y = 0, b. \tag{16}$$

We have $\partial T/\partial z = 0$ for $z = 0$ and therefore also for the region Γ_a and Γ_c. It can easily be verified that the function T_0 has the form

$$(17) \quad T_0(x,y,z) = \\ = \frac{2W}{ab\varkappa} \sum_{n,m}^{\infty} \frac{e^{-(z-\bar{\xi})\vartheta_{n,m}} + e^{-(z+\bar{\xi})\vartheta_{n,m}}}{\vartheta_{n,m}} \sin \alpha_n \bar{\xi} \sin \beta_m \bar{\eta} \sin \alpha_n x \sin \beta_m y,$$

where

$$\alpha_n = \frac{n\pi}{a}, \quad \beta_m = \frac{m\pi}{b}, \quad \vartheta_{n,m} = \sqrt{\alpha_n^2 + \beta_m^2}.$$

The solution of the Eq. (6) with the boundary conditions

$$G = 0 \quad \text{for} \quad x = 0, a \quad \text{and} \quad y = 0, b,$$

$$(18) \quad \left.\frac{\partial G}{\partial z}\right|_{z=0} = \delta(x-\xi)\,\delta(y-\eta),$$

is

$$(19) \quad G(x,y,z;\xi,\eta,0) = -\frac{4}{ab} \sum_{n,m}^{\infty} \frac{e^{-z\vartheta_{n,m}}}{\vartheta_{n,m}} \sin \alpha_n \xi \sin \beta_n \eta \sin \alpha_n x \sin \beta_m y.$$

The integral equation (9) takes the form

$$(20) \quad \int_{c_1}^{c_2}\int_{d_1}^{d_2} \psi(\xi,\eta) \sum_{n,m}^{\infty} \frac{\sin \alpha_n \xi \sin \beta_m \eta}{\vartheta_{n,m}} \sin \alpha_n x \sin \beta_m y = \\ = \frac{W}{\varkappa} \sum_{n,m}^{\infty} \frac{e^{-\bar{\xi}\vartheta_{nm}}}{\vartheta_{nm}} \sin \alpha_n \bar{\xi} \sin \beta_m \bar{\eta} \sin \alpha_n x \sin \beta_m y.$$

An accurate solution of this integral equation presents considerable difficulties. However, this equation can be solved in an approximate manner by replacing integrals by sums, thus reducing the solution of the integral equation to that of a system of linear algebraic equations. In the particular case of $c_1 = 0, c_2 = a$, $d_1 = 0$, $d_2 = b$ we obtain the solution of the Eq. (20) in the form of the trigonometric series

$$(21) \quad \psi(\xi,\eta) = \frac{4W}{ab\varkappa} \sum_{n,m}^{\infty} e^{-\bar{\xi}\vartheta_{nm}} \sin \alpha_n \xi \sin \beta_m \bar{\eta} \sin \alpha_n x \sin \beta_m y.$$

From the integral relation (3) we find that

$$(22) \quad T(x,y,z) = \\ = \frac{2W}{ab\varkappa} \sum_{n,m}^{\infty} \frac{\sin \alpha_n \bar{\xi} \sin \beta_m \bar{\eta}}{\vartheta_{nm}} \left[e^{-(z-\bar{\xi})\vartheta_{nm}} - e^{-(z+\bar{\xi})\vartheta_{mn}}\right] \sin \alpha_n x \sin \beta_m y.$$

Consider, in addition, the case, where no heat source exists inside the elastic body, the boundary condition on the surface Γ_a being $T(x, y, 0) = V = \text{const}$. The integral equation (13) takes the form

$$\int_{c_1}^{c_2}\int_{d_1}^{d_2} \psi(\xi,\eta) \sum_{n,m}^{\infty} \frac{\sin \alpha_n \xi \sin \beta_m \eta}{\vartheta_{nm}} \sin \alpha_n x \sin \beta_m y = -\frac{Vab}{4}. \tag{23}$$

In the particular case of $c_1 = 0$, $c_2 = a$, $d_1 = 0$, $d_2 = b$ the solution of the Eq. (23) is

$$\psi(\xi,\eta) = -\frac{16\,V}{a\,b} \sum_{n,m}^{\infty} \frac{\vartheta_{n,m}}{\alpha_n \beta_m} \sin \alpha_n x \sin \beta_m y. \tag{24}$$

The temperature field is obtained from the Eq. (3)

$$T(x,y,z) = \frac{16\,V}{ab} \sum_{n,m}^{\infty} \frac{e^{-z\vartheta_{nm}}}{\alpha_n \beta_m} \sin \alpha_n x \sin \beta_m y. \tag{25}$$

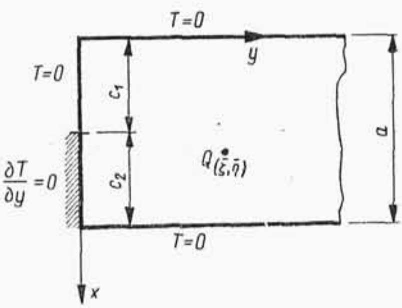

Fig. 3

(b) Consider a thin plate strip with a concentrated heat source at Q: Let $T = 0$ at the edges $x = 0$, $x = a$, and along the segment c_1 of the edge $y = 0$. Let $\partial T/\partial y = 0$ along the segment c_2 of the edge $y = 0$.

The functions $T_0(x, y)$, $G(x, y\ \xi, 0)$ take the form [2]:

$$T_0(x,y) = \frac{W}{a\varkappa} \sum_{n=1}^{\infty} \frac{\sin \alpha_n \xi}{\alpha_n} [e^{-\alpha_n(y-\bar{\eta})} + e^{-\alpha_n(y+\bar{\eta})}] \sin \alpha_n x, \tag{26}$$

$$T_0(x,y) = -\frac{W}{4\pi\varkappa} \ln \frac{\left[\cosh\frac{\pi}{a}(y-\bar{\eta}) - \cos\frac{\pi}{a}(x-\bar{\xi})\right]\left[\cosh\frac{\pi}{a}(y+\bar{\eta}) - \cos\frac{\pi}{a}(x-\bar{\xi})\right]}{\left[\cosh\frac{\pi}{a}(y-\bar{\eta}) - \cos\frac{\pi}{a}(x+\bar{\xi})\right]\left[\cosh\frac{\pi}{a}(y+\bar{\eta}) - \cos\frac{\pi}{a}(x+\bar{\xi})\right]},$$

$$G(x,y;\xi,0) = -\frac{2}{a} \sum_{n=1}^{\infty} \frac{e^{-\alpha_n y}}{\alpha_n} \sin \alpha_n \xi \sin \alpha_n x = \tag{27}$$

$$= \frac{1}{2\pi} \ln \frac{\cosh\frac{\pi}{a} y - \cos\frac{\pi}{a}(x-\xi)}{\cosh\frac{\pi}{a} y - \cos\frac{\pi}{a}(x+\xi)}.$$

In order to determine the function $\psi(\xi)$ appearing on the segment c_1 of the edge $y=0$ we should solve the integral equation (9):

$$\int_0^{c_1} \psi(\xi) \ln \frac{\left|\sin \frac{\pi}{2a}(x-\xi)\right|}{\sin \frac{\pi}{2a}(x+\xi)} d\xi = \frac{W}{\varkappa} \ln \frac{\cosh \frac{\pi}{a}\bar{\eta} - \cos \frac{\pi}{a}(x-\bar{\xi})}{\cosh \frac{\pi}{a}\bar{\eta} - \cos \frac{\pi}{a}(x-\bar{\xi})}. \tag{28}$$

In the particular case where $c_1 = a$, the solution of the Eq. (28) is the function

$$\psi(\xi) = \frac{W}{2a\varkappa} \sinh \frac{\pi \bar{\eta}}{a} \left[\frac{1}{\cosh \frac{\pi}{a}\bar{\eta} - \cos \frac{\pi}{a}(x-\bar{\xi})} - \frac{1}{\cosh \frac{\pi}{a}\bar{\eta} - \cos \frac{\pi}{a}(x+\bar{\xi})} \right] = \frac{2W}{a\varkappa} \sum_{n=1}^{\infty} e^{-\alpha_n \bar{\eta}} \sin \alpha_n \bar{\xi} \sin \alpha_n x. \tag{29}$$

Substituting $\psi(\xi)$ in the Eq. (3), we have

$$T(x,y) = \frac{W}{a\varkappa} \sum_{n=1}^{\infty} \frac{e^{-\alpha_n (y-\bar{\eta})} - e^{-\alpha_n (y+\bar{\eta})}}{\alpha_n} \sin \alpha_n \bar{\xi} \sin \alpha_n x, \tag{30}$$

or, in a closed form,

$$T(x,y) = \frac{W}{4\pi\varkappa} \ln \frac{\left[\cosh \frac{\pi}{a}(y+\bar{\eta}) - \cos \frac{\pi}{a}(x-\bar{\xi})\right]\left[\cosh \frac{\pi}{a}(y-\bar{\eta}) - \cos \frac{\pi}{a}(x+\bar{\xi})\right]}{\left[\cosh \frac{\pi}{a}(y+\bar{\eta}) - \cos \frac{\pi}{a}(x+\bar{\xi})\right]\left[\cosh \frac{\pi}{a}(y-\bar{\eta}) - \cos \frac{\pi}{a}(x-\bar{\xi})\right]}. \tag{31}$$

In the particular case of no heat source in the region of the plate, the boundary condition along the segment c_1 of the edge $y=0$ being $T=V=$ $=$ const., the integral equation (9) takes the form

$$\int_0^{c_1} \psi(\xi) \ln \frac{\left|\sin \frac{\pi}{2a}(x-\xi)\right|}{\sin \frac{\pi}{2a}(x+\xi)} d\xi = 2\pi V. \tag{32}$$

In the particular case where $c_1 = a$, we obtain the solution of this equation in the closed form

$$\psi(\xi) = -\frac{2V}{a}\left(\sin \frac{\pi x}{a}\right)^{-1}. \tag{33}$$

The temperature field will be found using the Eq. (3). We obtain

$$(34)\qquad T(x,y)=\frac{2V}{\pi}\operatorname{arctg}\frac{2e^{-\frac{\pi}{a}y}\sin\frac{\pi x}{a}}{1-e^{-2\pi y/a}}=\frac{4V}{a}\sum_{n=1}^{\infty}\frac{e^{-a_n y}}{a_n}\sin a_n x.$$

(c) Consider an elastic semi-space. Let $T=V(x,y)$ over a region Γ_a of the plane bounding the semi-space. Let $\partial T/\partial z|_{z=0}=0$ outside this region. The function $\partial T/\partial z|_{z=0}=\psi(\xi,\eta)$, appearing in the region Γ_a, will be found from the Eq. (13)

$$(35)\qquad \iint_{(\Gamma_a)}\psi(\xi,\eta)\,G(x,y,0;\xi,\eta,0)\,d\xi\,d\eta=V(x,y).$$

The kernel G of the integral equation should satisfy the equation $\nabla^2 G=0$ with the boundary conditions

$$(36)\qquad \left.\frac{\partial G}{\partial z}\right|_{z=0}=\delta(x-\xi)\,\delta(y-\eta),$$

$G=0$ at infinity.

It can easily be verified that the function G has the form

$$(37)\qquad G(x,y,z;\xi,\eta,0)=-\frac{1}{2\pi R}\quad R=[(x-\xi)^2+(y-\eta)^2+z^2]^{1/2}.$$

Therefore,

$$(35')\qquad \iint_{(\Gamma_a)}\frac{\psi(\xi,\eta)\,d\xi\,d\eta}{\sqrt{(x-\xi)^2+(y-\eta)^2}}=-2\pi V(x,y).$$

An equation of this type appears in the contact problems of the theory of elasticity, that is in problems of a rigid punch acting on an elastic semi-space. In these problems the function ψ corresponds to a function of the normal stress between the punch and the elastic body, and the function V to a function of vertical displacement. By applying this analogy a number of solutions of contact problems can be used for determining temperature fields *).

(d) Let a heat source of intensity W act at the point Q of the coordinates $(0,0,\zeta)$ in an elastic semi-space. Let $T=0$ over the region Γ_a in the $z=0$ plane bounding the semi-space, the condition for the external region being $\partial T/\partial z|_{z=0}=0$. The solution is sought in an integral form (3), where the function ψ is found from the solution of the integral equation (13)

$$(38)\qquad \iint_{(\Gamma_a)}\psi(\xi,\eta)\,G(x,y,0;\xi,\eta,0)\,d\xi\,d\eta+T_0(x,y,0)=0,$$

where $T_0(x,y,z)$ is the solution of the equation $\nabla^2 T=0$ with the boundary conditions $\partial T/\partial z|_{z=0}=0$ and $T_0=0$ at infinity.

*) An abundant literature on contact problems has been gathered in the monographs by L. A. Galin [3], and I. Y. Stayerman [4].

It can easily be verified that this equation, together with the boundary conditions, is satisfied by the function

$$T_0(x, y, z) = \frac{W}{4\pi\varkappa}\left(\frac{1}{R_1} + \frac{1}{R_2}\right), \tag{39}$$

where

$$R_{1,2} = [z \mp \zeta)^2 + x^2 + y^2]^{1/2}.$$

The Eq. (38) therefore takes the form

$$\iint\limits_{(\Gamma_a)} \frac{\psi(\xi, \eta)\, d\xi\, d\eta}{\sqrt{(x-\zeta)^2 + (y-\eta)^2}} = \frac{W}{\varkappa}(x^2 + y^2 + \zeta^2)^{-1/2}. \tag{40}$$

In the particular case, where Γ_a is a circular region of radius a, the integral equation (40) may be replaced by the equation

$$\int\limits_0^\infty \frac{\psi(\varrho)}{\sqrt{\varrho + r}} K\left(\frac{2\sqrt{\varrho r}}{\varrho + r}\right) d\varrho = \frac{W}{4\varkappa}(r^2 + \zeta^2)^{-1/2}, \tag{41}$$

where

$$K(\lambda) = \int\limits_0^{\pi/2} (1 - \lambda^2 \sin^2 \Phi)^{-1/2}\, d\Phi = \frac{\pi}{2}\, {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; \lambda^2),$$

is a complete elliptic integral.

DEPARTMENT OF MECHANICS OF CONTINUOUS MEDIA, INSTITUTE OF BASIC TECHNICAL PROBLEMS, POLISH ACADEMY OF SCIENCES

REFERENCES

[1] H. S. Carslaw, J. C. Jaeger, *Conduction of heat in solids*, Oxford, 1947.

[2] W. Nowacki, *The state of stress in a thin plate due to the action of sources of heat*, A. J. P. Ch., Mémoires, Zurich, 1952.

[3] L. A. Galin, *Contact problems of the theory of elasticity* (in Russian), Moscow, 1953.

[4] I. J. Stayerman, *The contact problem of the theory of elasticity* (in Russian), Moscow, 1949.