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APPLIED MECHANICS

State of Stress in an Infinite and Semi-Infinite Elastic Space Due to an Instantaneous Source of Heat

by

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1. An instantaneous source of heat in an infinite space

Let the heat quantity $Q = W\varrho c$ be produced at the point $A(\xi, \eta, \zeta)$ of an isotropic elastic space at the instant $t = 0$, where W denotes the intensity of the heat source, ϱ — the density and c — the specific heat of the elastic medium. The temperature field is described by the differential equation

$$\nabla^2 T = \frac{1}{k} \frac{\partial T}{\partial t}, \tag{1.1}$$

where $T(x, y, z, t)$ is the temperature at the point (x, y, z) and at the instant t, and $k = \lambda/\varrho c$, where λ is the coefficient of heat conduction (thermometric conductivity).

The temperature field due to the action of the concentrated instantaneous heat source at the point $A(\xi, \eta, \zeta)$ is determined by the equation, [1],

$$T = \frac{W}{8(\pi k t)^{3/2}} e^{-\frac{R^2}{4kt}}, \quad \text{where} \quad R^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2, \tag{1.2}$$

or by the integral expression

$$T = \frac{W}{\pi^3} \int_0^\infty \int_0^\infty \int_0^\infty \exp[-kt(\alpha^2+\beta^2+\gamma^2)] \times \\ \times \cos\alpha(x-\xi)\cos\beta(y-\eta)\cos\gamma z \, d\alpha \, d\beta \, d\gamma. \tag{1.3}$$

In order to determine the state of stress, it is convenient to use the so-called potential of thermoelastic strain Φ. It is related to the displacement components u, v, w by the relations

$$\frac{\partial \Phi}{\partial x} = u, \quad \frac{\partial \Phi}{\partial y} = v, \quad \frac{\partial \Phi}{\partial z} = w. \tag{1.4}$$

The relations (1.4) introduced in the three displacement equations of the theory of elasticity, can be reduced to the unique equation, [2],

$$(1.5) \qquad \nabla^2 \Phi = \frac{1+\nu}{1-\nu} \alpha_t T.$$

The Eq. (1.5) is valid if the dynamical effects are disregarded. It is known that the displacement accelerations are disregarded in the displacement equations of the theory of elasticity. Our problem is therefore considered to be quasi-statical.

In the Eq. (1.5) ν denotes Poisson's ratio and α_t the coefficient of thermal dilatation.

Differentiating the Eq. (1.5) with respect to time, and using the Eq. (1.1), we obtain

$$(1.6) \qquad \nabla^2 \left(\frac{\partial \Phi}{\partial t}\right) = \frac{1+\nu}{1-\nu} \alpha_t k \nabla^2 T.$$

Hence,

$$(1.7) \qquad \Phi = \frac{1+\nu}{1-\nu} \alpha_t k \int T dt.$$

From the relations (1.1) and (1.7) the following relation is obtained:

$$(1.8) \qquad \nabla^2 \Phi = \frac{1}{k} \frac{\partial \Phi}{\partial t}.$$

The stress components are connected with the potential of thermoelastic strain by the relations, [2],

$$(1.9) \qquad \begin{aligned}
\overline{\sigma}_{xx} &= -2G\left(\nabla^2\Phi - \frac{\partial^2 \Phi}{\partial x^2}\right) = 2G\left(\frac{\partial^2 \Phi}{\partial x^2} - \frac{1}{k}\frac{\partial \Phi}{\partial t}\right),\\
\overline{\sigma}_{yy} &= -2G\left(\nabla^2\Phi - \frac{\partial^2 \Phi}{\partial y^2}\right) = 2G\left(\frac{\partial^2 \Phi}{\partial y^2} - \frac{1}{k}\frac{\partial \Phi}{\partial t}\right),\\
\overline{\sigma}_{zz} &= -2G\left(\nabla^2\Phi - \frac{\partial^2 \Phi}{\partial z^2}\right) = 2G\left(\frac{\partial^2 \Phi}{\partial z^2} - \frac{1}{k}\frac{\partial \Phi}{\partial t}\right),\\
\overline{\sigma}_{xy} &= 2G\frac{\partial^2 \Phi}{\partial x \partial y}, \quad \overline{\sigma}_{yz} = 2G\frac{\partial^2 \Phi}{\partial y \partial z}, \quad \overline{\sigma}_{zx} = 2G\frac{\partial^2 \Phi}{\partial z \partial x},
\end{aligned}$$

G denoting the modulus of elasticity in shear.

In order to determine the stress components we shall proceed as follows. We determine a particular integral of the Eq. (1.5) and substitute it in the relations (1.9). This integral does not satisfy, in general, every boundary condition. In the latter case to the state of stress $(\overline{\sigma})$ a state of stress $(\overline{\overline{\sigma}})$ will be superposed, chosen in a way to satisfy every boundary condition. The resulting stresses will be obtained by adding the corresponding stress components pertaining to the states $(\overline{\sigma})$ and $(\overline{\overline{\sigma}})$.

It is seen that according to the Eqs. (1.2) and (1.7) the particular integral of the Eq. (1.5) will take the form

$$\Phi = \frac{1+\nu}{1-\nu}\,\alpha_t\,\frac{Wk}{8\pi^{3/2}}\int_0^t (kt)^{-3/2}\exp\left(-\frac{R^2}{4kt}\right)dt \tag{1.10}$$

or

$$\Phi = -\frac{1+\nu}{1-\nu}\,\alpha_t\,\frac{W}{\pi^3}\int_0^\infty\int_0^\infty\int_0^\infty (\alpha^2+\beta^2+\gamma^2)^{-1}\exp\left[-kt(\alpha^2+\beta^2+\gamma^2)\right]\times \\ \times\cos\alpha(x-\xi)\cos\beta(y-\eta)\cos\gamma(z-\zeta)\,d\alpha\,d\beta\,d\gamma. \tag{1.11}$$

By substituting $u = R^2/4\,kt$, the integral (1.10) can be reduced to

$$\Phi = \frac{1+\nu}{1-\nu}\,\alpha_t\,\frac{W}{4\pi^{3/2}R}\int_u^\infty \frac{e^{-u}}{\sqrt{u}}\,du. \tag{1.12}$$

Since

$$\int_u^\infty u^{-1/2}\exp(-u)\,du = \sqrt{\pi}\,[1-\operatorname{erf}(\sqrt{u})] = \sqrt{\pi}\operatorname{erfc}\sqrt{u},$$

we have

$$\Phi = \frac{1+\nu}{1-\nu}\,\frac{\alpha_t W}{4\pi}\cdot R^{-t}\operatorname{erfc}\left(\frac{R}{2\sqrt{kt}}\right). \tag{1.13}$$

Substituting the particular integral Φ in the relations (1.9) and introducing the notation

$$N = \frac{1+\nu}{1-\nu}\,\alpha_t\,\frac{GW}{2\pi},$$

we have

$$\bar\sigma_{xx} = -\frac{N}{R^3}\left\{\left[1-\frac{3(x-\xi)^2}{R^2}\right]\left[\operatorname{erfc}\left(\frac{R}{2\sqrt{kt}}\right)+\frac{R}{\sqrt{kt}}\exp\left(\frac{-R^2}{4kt}\right)\right]+\right. \\ \left.+\frac{R\exp\left(-\frac{R^2}{4kt}\right)}{2\sqrt{\pi}(kt)^{3/2}}\left[R^2-(x-\xi)^2\right]\right\} \tag{1.14 a}$$

and two analógous equations for $\bar\sigma_{yy}$, $\bar\sigma_{zz}$;

$$\bar\sigma_{xy} = \frac{3N}{R^5}(x-\xi)(y-\eta)\left[\operatorname{erfc}\left(\frac{R}{2\sqrt{kt}}\right)+\right. \\ \left.+\frac{R}{\sqrt{\pi kt}}\exp\left(-\frac{R^2}{4kt}\right)\left(1+\frac{1}{6}\frac{R^2}{kt}\right)\right] \tag{11.4 b}$$

and two analogous equations for σ_{xz} and $\bar\sigma_{zy}$.

It is seen that the normal and shear stresses vanish at infinity at every instant t.

They do not vanish, however, for $t = \infty$ for arbitrarily chosen x, y, z.
For $t = \infty$ we have

$$\sigma_{xx,\infty} = -\frac{N}{R^3}\left[1 - \frac{3(x-\xi)}{R^2}\right] \tag{1.15 a}$$

and two analogous equations for $\sigma_{yy,\infty}$, $\sigma_{zz,\infty}$, and

$$\sigma_{xy,\infty} = \frac{3N}{R^5}(x-\xi)(y-\eta) \tag{1.15 b}$$

and two analogous equations for $\sigma_{xz,\infty}$, $\sigma_{yx,\infty}$.

If these values of stresses are substracted from the stresses expressed by the Eqs. (1.14a) and (1.14b), we obtain the stress components satisfying all conditions at infinity.

We obtain finally:

$$\sigma_{xx} = -\frac{N}{R^3}\left\{\left[1 - \frac{3(x-\xi)^2}{R^2}\right]\left[\frac{R}{\sqrt{kt}}\exp\left(-\frac{R^2}{4kt}\right) - \operatorname{erf}\left(\frac{R}{2\sqrt{kt}}\right)\right] + \right.$$
$$\left. + \frac{R\exp\left(-\frac{R^2}{4kt}\right)}{2\sqrt{\pi}(kt)^{3/2}}\left[R^2 - (x-\xi)^2\right]\right\} \tag{1.16 a}$$

and two analogous equations for σ_{yy}, σ_{zz}; and

$$\sigma_{xy} = \frac{3N}{R^5}(x-\xi)(y-\eta)\left[\frac{R}{\sqrt{\pi kt}}\exp\left(-\frac{R^2}{4kt}\right)\left(1 + \frac{R^2}{6kt}\right) - \right.$$
$$\left. - \operatorname{erf}\left(\frac{R}{2\sqrt{kt}}\right)\right] \tag{1.16 b}$$

and two analogous equations for σ_{xz}, σ_{yz}.

Equations for stress components in polar co-ordinates are particularly simple in the case of spherical symmetry of strain. Let us assume the heat source to be located at the origin. Then,

$$\Phi = \frac{2N}{G}R^{-1}\operatorname{erfc}\left(\frac{R}{2\sqrt{kt}}\right), \quad \text{where} \quad R = \sqrt{x^2 + y^2 + z^2}. \tag{1.17}$$

The stress components are expressed by

$$\begin{aligned} \sigma_{RR} &= 2G\left(\varepsilon_{RR} + \frac{\nu}{1-2\nu}\Theta - \frac{1+\nu}{1-2\nu}a_t T\right) \\ \sigma_{\varphi\varphi} = \sigma_{\vartheta\vartheta} &= 2G\left(\varepsilon_{\varphi\varphi} + \frac{\nu}{1-2\nu}\Theta - \frac{1+\nu}{1-2\nu}a_t T\right) \\ \sigma_{R\varphi} = 0, &\quad \sigma_{\varphi\vartheta} = 0, \quad \sigma_{R\vartheta} = 0. \end{aligned} \tag{1.18}$$

The notations are as follows:

$$\varepsilon_{RR}=\frac{du_R}{dR},\quad \varepsilon_{\varphi\varphi}=\frac{u_R}{R},$$

(1.19)

$$\Theta=\varepsilon_{RR}+\varepsilon_{\vartheta\vartheta}+\varepsilon_{\varphi\varphi}=\frac{du_R}{dR}+2\frac{u_R}{R},\quad u_R=\frac{d\Phi}{dR},$$

where u_R is the radial displacement.

Bearing in mind that for $t=\infty$ the stresses should be zero, we obtain the Eqs. (1.18) in the form

$$\sigma_{RR}=-\frac{8N}{R^3}\left[\operatorname{erf}\frac{R}{2\sqrt{kt}}-\frac{2R}{\sqrt{\pi kt}}\exp\left(-\frac{R^2}{4kt}\right)\right],$$

(1.20) $$\sigma_{\varphi\varphi}=\sigma_{\vartheta\vartheta}=\frac{4N}{R^3}\left[\operatorname{erf}\frac{R}{2\sqrt{kt}}-\frac{R}{\sqrt{\pi kt}}\exp\left(-\frac{R^2}{4kt}\right)\left(1+\frac{R^2}{2kt}\right)\right]$$

$$\sigma_{R\varphi}=0,\quad \sigma_{R\vartheta}=0,\quad \sigma_{\vartheta\varphi}=0.$$

Figs. 1a, b, c represent diagrams of T, σ_{RR}, $\sigma_{\varphi\varphi}$, in function of the radius R for a few values of the parameter $\vartheta=4kt$.

If in the Eqs. (1.16a), (1.16b), we put $W=1$, these equations represent the Green functions of our problem. The knowledge of these functions enables the solution of the more general problem of determining the stress components σ^*_{ij} at any point $B(x,y,z)$ due to the action of instantaneous heat sources $w(\xi,\eta,\zeta)$ distributed over a finite region of the elastic space.

Using the superposition method, we obtain the σ^*_{ij} components from the equations

(1.21) $$\sigma^*_{ij}(x,y,z,t)=\iiint_{(V)} w(\xi,\eta,\zeta)\,\sigma_{ij}(x,y,z;\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta\quad i,j=x,y,z.$$

2. An instantaneous source of heat in a semi-infinite space

Consider an instantaneous source of heat of intensity W acting at a point $A(0,0,\zeta)$ of a semi-infinite space. Our problem is to determine the stress components when the plane $z=0$, bounding the semi-infinite space, is free from stresses.

In addition, we require that $T=0$ for $z=0$. Thus, the boundary conditions of our problem are

(2.1) $$\sigma_{zz}=0,\quad \sigma_{zx}=0,\quad \sigma_{zy}=0,\quad T=0,\quad \text{for}\quad z=0.$$

The first and the last conditions will be satisfied if a positive and a negative sources of heat are located at the points $A(0,0,\zeta)$ and $A'(0,0,-\zeta)$ of the infinite space, respectively.

For such a system of heat sources, antisymmetric in relation to the plane $z = 0$, we obtain, using the Eq. (1.13),

$$\Phi = \frac{1+\nu}{1-\nu}\,\frac{a_t W}{4\pi}\left[R_1^{-1}\operatorname{erfc}\left(\frac{R_1}{2\sqrt{kt}}\right) - R_2^{-1}\operatorname{erfc}\left(\frac{R_2}{2\sqrt{kt}}\right)\right], \tag{2.2}$$

where

$$R_{1,2} = x^2 + y^2 (z \mp \zeta)^2.$$

For further considerations it will be convenient to represent the function by the Fourier integral

$$\Phi = -\frac{1+\nu}{1-\nu}\,\frac{a_t W}{\pi^3}\int_0^\infty\int_0^\infty\int_0^\infty (\alpha^2+\beta^2+\gamma^2)^{-1}\exp\left[-kt(\alpha^2+\beta^2+\gamma^2)\right]\times$$
$$\times \cos\alpha x\cos\beta y\left[\cos\gamma(z-\zeta) - \cos\gamma(z+\zeta)\right]d\alpha\, d\beta\, d\gamma. \tag{2.3}$$

The stress components due to the action of two sources, antisymmetric in relation to the plane $z = 0$, can easily be found by means of the Eqs. (1.14a), (1.14b).

Thus, for instance,

$$\bar{\sigma}_{xx} = -\frac{N}{R_1^3}\left\{\left[1-\frac{3x^2}{R_1^2}\right]\left[\operatorname{erfc}\left(\frac{R_1}{2\sqrt{kt}}\right) + \frac{R_1}{\sqrt{kt}}\exp\left(\frac{-R_1^2}{4kt}\right)\right] + \right.$$
$$\left. + \frac{R_1\exp\left(-\frac{R_1^2}{4kt}\right)}{2\sqrt{\pi}(kt)^{3/2}}\left[R_1^2 - x^2\right]\right\} + \frac{N}{R_2^3}\left\{\left(1-\frac{3x^2}{R_2^2}\right)\left[\operatorname{erfc}\left(\frac{R_2}{2\sqrt{kt}}\right) + \right.\right.$$
$$\left.\left. + \frac{R_2}{\sqrt{kt}}\exp\left(-\frac{R_2^2}{4kt}\right)\right] + \frac{R_2\exp\left(-\frac{R_2^2}{4kt}\right)}{2\sqrt{\pi}(kt)^{3/2}}\left[R_2^2 - x^2\right]\right. \tag{2.4 a}$$

$$\bar{\sigma}_{xy} = 3Nxy\left\{\frac{1}{R_1^5}\left[\operatorname{erfc}\left(\frac{R_1}{2\sqrt{kt}}\right) + \frac{R_1}{kt)^{1/2}}\exp\left(-\frac{R_1^2}{4kt}\right)\cdot\left(1+\frac{1}{6}\frac{R_1^2}{kt}\right)\right] - \right.$$
$$\left. - \frac{1}{R_2^5}\left[\operatorname{erfc}\left(\frac{R_2}{2\sqrt{kt}}\right) + \frac{R_2}{(\pi kt)^{1/2}}\exp\left(-\frac{R_2^2}{4kt}\right)\cdot\left(1+\frac{1}{6}\frac{R_2^2}{kt}\right)\right]\right\}. \tag{2.4 b}$$

The function Φ does not satisfy all boundary conditions. In the plane $z = 0$ the stresses $\bar{\sigma}_{xz}$, $\bar{\sigma}_{yz}$ do not vanish. In order to suppress these stresses an additional problem must be solved, and the state of stress $(\bar{\bar{\sigma}}_{ij})$ in the elastic semi-infinite space for $T = 0$, due to the action of the shear stresses $-\bar{\sigma}_{xz}$, $-\bar{\sigma}_{yz}$ in the plane $z = 0$, must be determined. The conditions of this problem are

$$\bar{\sigma}_{xz} + \bar{\bar{\sigma}}_{xz} = 0, \quad \bar{\sigma}_{yz} + \bar{\bar{\sigma}}_{yz} = 0, \quad \bar{\bar{\sigma}}_{zz} = 0 \quad \text{for} \quad z = 0. \tag{2.5}$$

The stresses σ_{ij}, due to the action of a heat source at the point $A(0, 0, \zeta)$ of the elastic semi-infinite space, will be obtained by superposition of the stresses $\bar{\sigma}_{ij}$ and $\bar{\bar{\sigma}}_{ij}$.

In order to determine the stress components $(\bar{\bar{\sigma}}_{ij})$ in the semi-infinite space we use the Galerkin displacement function, [3].

This function reduces the system of three differential equations of displacement to the unique biharmonic equation

$$\nabla^2 \nabla^2 \varphi = 0, \tag{2.6}$$

where the stress components $(\bar{\bar{\sigma}}_{ij})$ are expressed by the relations

$$\begin{aligned}
&\bar{\bar{\sigma}}_{xx} = \frac{\partial}{\partial z}\left(\nu \nabla^2 \varphi - \frac{\partial^2 \varphi}{\partial x^2}\right), \qquad \sigma_{yy} = \frac{\partial}{\partial z}\left(\nu \nabla^2 \varphi - \frac{\partial^2 \varphi}{\partial y^2}\right),\\
&\sigma_{zz} = \frac{\partial}{\partial z}\left[(1-\nu)\nabla^2 \varphi + \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}\right], \qquad \sigma_{xy} = -\frac{\partial^3 \varphi}{\partial x \, \partial y \, \partial z},\\
&\sigma_{xz} = \frac{\partial}{\partial x}\left[\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} - \nu \nabla^2 \varphi\right], \qquad \bar{\bar{\sigma}}_{yz} = \frac{\partial}{\partial y}\left[\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} - \nu \nabla^2 \varphi\right].
\end{aligned} \tag{2.7}$$

The displacement function will be assumed in the form of the Fourier integral

$$\varphi = \int_0^\infty \int_0^\infty Z(\alpha, \beta, z) \cos \alpha x \cos \beta y \, d\alpha \, d\beta, \tag{2.8}$$

where

$$Z = (A + B \delta z) e^{-\delta z}, \qquad \delta = \sqrt{\alpha^2 + \beta^2},$$

and A, B are functions of the parameters α, β.

The third of the boundary conditions (2.5) leads (as can easily be seen by substituting (2.8) in σ_{zz}, obtained from (2.7)) to the relation

$$(2-\nu) Z'(0) \delta^2 - (1-\nu) Z'''(0) = 0. \tag{2.9}$$

Since $Z'(0) = (B - A)\delta$, $\quad Z'''(0) = (3B - A)\delta^3$, we have

$$B = -A \frac{1}{1 - 2\nu}. \tag{2.10}$$

The first two boundary conditions (2.5) can be represented in the form

$$\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}\left[2G \frac{\partial \Phi}{\partial z} + \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} - \nu \nabla^2 \varphi\right]_{z=0} = 0. \tag{2.11}$$

It is clear that these conditions can be reduced to a single condition:

$$\left[2G \frac{\partial \Phi}{\partial z} + \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} - \nu \nabla^2 \varphi\right]_{z=0} = 0. \tag{2.12}$$

Expressing the function Φ by the Eq. (2.3) and the function φ by the Eq. (2.8), we obtain

$$(2.13)\qquad -\frac{4W(1+\nu)}{\pi^3(1-\nu)}\alpha_t e^{-kt\delta^2}\int_0^\infty \frac{\gamma\, e^{-kt\gamma^2}}{\gamma^2+\delta^2}\sin\gamma\zeta\, d\gamma-(1-\nu)Z(0)\delta^2-\nu Z''(0)=0.$$

Since $Z(0)=A$, $Z''(0)=-(2B-A)\delta^2$, we obtain from the Eqs. (2.13) and (2.10)

$$(2.14)\qquad A=-\frac{4W(1+\nu)(1-2\nu)\alpha_t}{\pi^3(1-\nu)\delta^2}e^{-kt\delta^2}\int_0^\infty \frac{\gamma\exp(-kt\gamma^2)}{\gamma^2+\delta^2}\sin\gamma\zeta\, d\gamma;$$

$$B=-\frac{A}{1-2\nu}.$$

Since

$$(2.15)\qquad \int_0^\infty \frac{\gamma\exp(-kt\gamma^2)}{\delta^2+\gamma^2}\sin\gamma\zeta\, d\gamma=\frac{\pi}{4}\exp(kt\delta^2)\times$$

$$\times\left[\exp(-\delta\zeta)\operatorname{erfc}\left(\delta\sqrt{kt}-\frac{\zeta}{2\sqrt{kt}}\right)-\exp(\delta\zeta)\operatorname{erfc}\left(\delta\sqrt{kt}+\frac{\zeta}{2\sqrt{kt}}\right)\right],$$

we have

$$(2.16)\qquad \varphi=-\frac{(1+\nu)(1-2\nu)\alpha_t W}{\pi^2(1-\nu)}\int_0^\infty\int_0^\infty \frac{1}{\delta^2}\left(1-\frac{1}{1-2\nu}\delta z\right)\times$$

$$\times\left[\exp(-\delta(z+\zeta))\operatorname{erfc}\left(\delta\sqrt{kt}-\frac{\zeta}{2\sqrt{kt}}\right)-\right.$$

$$\left.-\exp(-\delta(z-\xi))\operatorname{erfc}\left(\delta\sqrt{kt}+\frac{\zeta}{2\sqrt{kt}}\right)\right]\cos\alpha x\cos\beta y\, d\alpha\, d\beta.$$

The knowledge of the function φ enables us to determine the stress components $(\bar{\sigma}_{ij})$ from the Eqs. (2.7).

Since neither the function φ nor its third derivatives can be expressed in a closed form, the determination of the stress components $(\bar{\sigma}_{ij})$ can only be achieved by tedious numerical methods.

Consider an instantaneous action of a heat source at the point $A(0, 0, \zeta)$ in the case, where $u=0$, $v=0$, $w=0$ in the plane $z=0$.

Proceeding as in the previous case, we verify easily, assuming Φ according to the Eq. (2.2), that in the plane $z=0$

$$(2.17)\qquad \bar{u}=\left.\frac{\partial\Phi}{\partial x}\right|_{z=0}=0,\qquad \bar{v}=\left.\frac{\partial\Phi}{\partial y}\right|_{z=0}=0.$$

The displacements $\overline{w}$ are different from zero. The state $(\overline{\overline{\sigma}}_{ij})$ should be superposed over the state $(\overline{\sigma}_{ij})$, the components of which are expressed by the Eqs. (2.4a), (2.4b).

The state under consideration concerns a semi-infinite elastic space for $T = 0$ subjected to the displacement $-\overline{w}$ in the plane $z = 0$. For this additional problem the boundary conditions are

$$\text{(2.18)} \qquad \overline{\overline{u}}|_{z=0} = 0, \quad \overline{\overline{v}}|_{z=0} = 0, \quad \overline{w} + \overline{\overline{w}}|_{z=0} = 0.$$

Since the displacement components are related to the function φ by the equations

$$\text{(2.19)} \qquad \begin{gathered} \overline{\overline{u}} = -\frac{1+\nu}{E}\frac{\partial^2 \varphi}{\partial x\, \partial z}, \qquad \overline{\overline{v}} = -\frac{1+\nu}{E}\frac{\partial^2 \varphi}{\partial y\, \partial z}, \\ w = \frac{1+\nu}{E}\left[\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + (1-2\nu)\nabla^2 \varphi\right], \end{gathered}$$

our problem is reduced to the solution of the differential equation (2.6) with the boundary conditions *)

$$\text{(2.20)} \qquad \frac{\partial \varphi}{\partial z}\bigg|_{z=0} = 0, \quad \left\{\frac{\partial \Phi}{\partial z} + \frac{1+\nu}{E}\left[\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + (1-2\nu)\nabla^2 \varphi\right]\right\}_{z=0} = 0.$$

The function φ is assumed here in the form (2.8).

The quantities $A, B,$ are found from the boundary conditions (2.20), the stress components $(\overline{\overline{\sigma}}_{ij})$ from the Eqs. (2.4a), (2.4b).

The final form of the stresses is determined by the relation $\sigma_{ij} = \overline{\sigma}_{ij} + \overline{\overline{\sigma}}_{ij}$.

The case where the plane $z = 0$ is free from shear stresses, and the displacements w are equal to zero in that plane, is equally easy. In determining the state of stress $(\overline{\overline{\sigma}}_{ij})$ the following boundary conditions should be assumed for the function φ:

$$\text{(2.21)} \qquad \begin{gathered} \left|\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} - \nu \nabla^2 \varphi + 2G\frac{\partial \Phi}{\partial z}\right|_{z=0} = 0, \\ \left|\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + (1-2\nu)\nabla^2 \varphi\right|_{z=0} = 0. \end{gathered}$$

The function Φ is expressed here by the Eq. (2.2) and the function φ by the Eq. (2.8).

If in all these cases the heat source is shifted from the point $A(0, 0, \zeta)$ to the point $A(\xi, \eta, \zeta)$; $x-\xi$, $y-\eta$ must be substituted for x, y in all the equations of this article. If, in addition, we assume that $W = 1$, the

*) In the Eqs. (2.19) and (2.20), E denotes Young's modulus.

stress components (σ_{ij}) will be the Green functions of our problem. Using these functions, we can obtain by integrating, according to the Eq. (1.15), the stress components $(\sigma_{ij}{}^*)$, due to the action of heat sources distributed over a finite region Γ of a semi- infinite elastic space.

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