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STRESS PROPAGATION IN AN INFINITE VISCOELASTIC BODY PRODUCED BY A TIME-VARIABLE POINT FORCE

WITOLD NOWACKI (WARSZAWA)

1. General Equations

The problem of propagation of elastic waves in a perfectly elastic infinite body due to the action of point forces has been dealt with in numerous papers. In 1956 G. Eason, J. Fulton and I. N. Sneddon, [1], gave a general method for solving this problem using Fourier's quadruple expotential transformation.

The object of the present paper is to determine the displacements and stresses due to the action of point forces and a centre of pressure in an infinite viscoelastic space.

We shall consider visco-elastic bodies in which the stress-strain relations are given by the equations, [2], [3]:

$$(1.1)\quad P_1(D)P_3(D)\sigma_{ij}^{(1)}=P_2(D)P_3(D)\varepsilon_{ij}^{(1)}+\delta_{ij}\frac{1}{3}[P_1(D)P_4(D)-P_2(D)P_3(D)]\varepsilon_{kk}^{(1)},$$

$$(1.2)\quad \sigma_{ij}^{(2)}=2\int_0^t a(t-\tau)\frac{\partial}{\partial\tau}\varepsilon_{ij}^{(2)}\,d\tau+\delta_{ij}\int_0^t b(t-\tau)\frac{\partial\varepsilon_{kk}^{(2)}}{\partial\tau}\,d\tau.$$

The linear operators $P_i(D)$ $i=1,2,3,4$ are expressed by the equations:

$$(1.3)\quad P_i(D)=\sum_{n=0}^{Ni}a_i^{(n)}D^{(n)},\qquad a_i^{(Ni)}\neq 0,$$

where $D^n=\partial^n/\partial t^n$ denotes the n-th derivative with respect to time. In the particular case of a perfectly elastic body, the operators $P_i(D)$ reduce to the first term of the sum (1.3)

$$a_1^{(0)}=1,\qquad a_2^{(0)}=2\mu_0,\qquad a_3^{(0)}=1,\qquad a_4^{(0)}=3\lambda_0+2\mu_0,$$

where λ_0, μ_0 are Lamé's constants.

The relation (1.2) was given by M. A. Biot, [4], and generalized by D. S. Berry, [3], to three-dimensional problems. $a(t)$, $b(t)$ are relaxation functions which for a perfectly elastic body reduce to Lamé's constants. Introducing the relations (1.1), (1.2) in the equations of motion, and expressing the strains in function of the displacements, we obtain the following displacement equation:

$$(1.4)\quad P_2(D)\,P_3(D)\,u^{(1)}_{i,kk} + \frac{1}{3}\,[2\,P_4(D)\,P_1(D) + P_2(D)\,P_3(D)]\,u^{(1)}_{k,ki} + \\ + 2\,P_3(D)\,P_1(D)\,F_i - 2\,P_1(D)\,P_3(D)\,\varrho \ddot{u}^{(1)}_i = 0,$$

$$(1.5)\quad \int_0^t \left\{a(t-\tau)\frac{\partial}{\partial\tau}\,u^{(2)}_{i,kk} + [b(t-\tau) + a(t-\tau)]\,\frac{\partial}{\partial\tau}\,u^{(2)}_{k,ki}\right\}d\tau + F_i - \varrho \ddot{u}^{(2)}_i = 0,$$

where F_i denote the mass force components.

Let us assume that the visco-elastic body was at the initial moment $(t \leqslant 0)$ free, that is, it was in the natural, unstressed state. Performing on the Eqs. (1.1) and (1.2) the Laplace transformation we obtain:

$$(1.6)\quad \sigma^*_{ij}(x_r, p) = \lambda^*(p)\,\varepsilon^*_{kk}(x_r, p)\,\delta_{ij} + 2\,\mu^*\,\varepsilon^*_{ij}(x_r,p), \quad f^*(x_r, p) = \int_0^\infty e^{-pt} f(x_r, t)\,dt,$$

with the following notations

$$(1.7)\qquad \lambda^*(p) = \frac{P_1(p)\,P_4(p) - P_2(p)\,P_3(p)}{3\,P_1(p)\,P_3(p)}, \qquad \mu^*(p) = \frac{P_2(p)}{2\,P_1(p)}$$

for a visco-elastic body for which the Eqs. (1.1) are valid and

$$(1.8)\qquad \lambda^*(p) = p b^*(p), \qquad \mu^*(p) = p a^*(p)$$

for a body where the Eqs. (1.2) hold.

Performing the Laplace transformation on the Eqs. (1.4) and (1.5), we shall express them by the unique expression:

$$(1.9)\qquad (\lambda^* + \mu^*)\,u^*_{k,ki} + \mu^*\,u^*_{i,kk} - \varrho p^2 u^*_i + F^*_i = 0 \qquad (i = 1, 2, 3).$$

Let us introduce the notations:

$$\frac{\lambda^* + 2\,\mu^*}{\mu^*} = \beta^2, \qquad \delta^2 = \beta^2 - 1 = \frac{\lambda^* + \mu^*}{\mu^*}, \qquad \eta = \frac{1}{\mu^*}, \qquad \sigma^2 = \frac{\varrho}{\mu^*}.$$

The system of equations (1.9) then takes the form:

$$(1.10)\qquad \delta^2\,u^*_{k,ki} + u^*_{i,kk} + F_i\,\eta - p^2\,\sigma^2\,u^*_i = 0 \qquad (i = 1, 2, 3).$$

To proceed further. We perform triple Fourier transformation on the Eqs. (1.10) (in the most general three-dimensional case), the inverse transformation giving the displacement expressed by triple Fourier integrals. This procedure is analogous to that chosen by the authors of Ref. [1], the insignificant difference being that the expotential Fourier transformation is replaced by the sine and cosine Fourier transformation. Next, after determining the Fourier integrals, we shall perform on the displacements the inverse Laplace transformation. Accurate solution is given for a Biot body only.

2. The Two-Dimensional and One-Dimensional Problem

Let a load $F = F_0(t)\,\delta(x_1)\,\delta(x_2)$, uniformly distributed along the x_1-axis, act parallelly to the x_3-axis. In this particular case, we have $u_3 = 0$, and the derivatives of the displacements u_1, u_2 in the x_3-direction are zero. The system of equations (1.10) is reduced to two equations:

$$\delta^2 u^*_{k,ki} + u^*_{i,kk} + \eta F_i\,\delta_{1i} - p^2\sigma^2 u^*_i = 0 \qquad (i = 1, 2), \tag{2.1}$$

where $F^* = F_0(p)\,\delta(x_1)\,\delta(x_2)$.

In order to solve the system of Eq. (2.1), we introduce a double Fourier integral defined by the relations:

$$\left\{\begin{aligned} u^*_i(x_1, x_2, p) &= \int_0^\infty\int_0^\infty \bar{u}_i(a_1, a_2, p)\cos a_1 x_1 \cos a_2 x_2\, da_1\, da_2 \qquad (i = 1, 2),\\ F^*(x_1, x_2, p) &= \int_0^\infty\int_0^\infty \bar{F}_0(a_1, a_2, p)\cos a_1 x_1 \cos a_2 x_2\, da_1\, da_2. \end{aligned}\right. \tag{2.2}$$

We have taken into consideration the fact that the displacement u^*_1 is symmetric in relation to the planes $x_1 = 0$, $x_2 = 0$, and that the displacement u^*_2 is antisymmetric in relation to these planes. The load F^* is symmetric in relation to both planes.

Substituting the integrals (2.2) in the system of equations (2.1), we obtain:

$$\left\{\begin{aligned} &(a_1^2\beta^2 + a_2^2 + p^2\sigma^2)\,\bar{u}_1 - a_1 a_2\,\delta^2\bar{u}_2 = \eta\bar{F},\\ &-a_1 a_2\,\delta^2\bar{u}_1 + (a_2^2\beta^2 + a_1^2 + p^2\sigma^2)\,\bar{u}_2 = 0. \end{aligned}\right. \tag{2.3}$$

Solving this system for $\bar{u}_1$ and $\bar{u}_2$, we obtain:

$$\left\{\begin{aligned} \bar{u}_1 &= \frac{\eta\bar{F}}{\beta^2}\,\frac{a_1^2 + \beta^2 a_2^2 + p^2\sigma^2}{(a_1^2 + a_2^2 + p^2\sigma^2)(a_1^2 + a_2^2 + p^2\tau^2)}, \qquad \tau = \frac{\sigma}{\beta},\\ \bar{u}_2 &= \frac{\eta\bar{F}\delta^2}{\beta^2}\,\frac{a_1\, a_2}{(a_1^2 + a_2^2 + p^2\sigma^2)(a_1^2 + a_2^2 + p^2\tau^2)}, \qquad \bar{F} = \frac{1}{\pi^2}\,F_0(p). \end{aligned}\right. \tag{2.4}$$

Performing the integrations prescribed by (2.2) and bearing in mind that

$$I^*(x_1, x_2, p; \sigma) = \int_0^\infty\int_0^\infty \frac{\cos a_1 x_1 \cos a_2 x_2}{a_1^2 + a_2^2 + p^2\sigma^2}\, da_1\, da_2 = \frac{\pi}{2}\,K_0(rp\sigma), \qquad r = (x_1^2 + x_2^2)^{1/2}, \tag{2.5}$$

we obtain the following displacements:

$$\left\{\begin{aligned} u^*_1 &= \frac{\eta F_0(p)}{2\beta^2\pi}\left\{K_0(rp\tau) - \frac{\beta^2}{p^2\sigma^2}\,\frac{\partial^2}{\partial x_2^2}\left[K_0(rp\tau) - K_0(rp\sigma)\right]\right\},\\ u^*_2 &= \frac{\eta F_0(p)}{2p^2\sigma^2\pi}\,\frac{\partial^2}{\partial x_1\,\partial x_2}\left[K_0(rp\tau) - K_0(rp\sigma)\right]. \end{aligned}\right. \tag{2.6}$$

The function $K_0(rp\sigma)$ appearing in these equations is a modified Bessel function of the third kind. For the stress components we obtain:

$$
(2.7)\quad \begin{cases}
\sigma_{11}^* + \sigma_{22}^* = \eta \dfrac{F_0(p)(\lambda^* + \mu^*)}{\beta^2 \pi} \dfrac{\partial}{\partial x_1} K_0(rp\sigma), \\[2ex]
\sigma_{11}^* - \sigma_{22}^* = \eta \dfrac{F_0(p)\mu^*}{\beta^2 \pi} \dfrac{\partial}{\partial x_1} \left\{ K_0(rp\tau) - \dfrac{2\beta^2}{p^2\sigma^2} \dfrac{\partial^2}{\partial x_2^2} [K_0(rp\tau) - K_0(rp\sigma)] \right\}, \\[2ex]
\sigma_{12}^* = \eta \dfrac{F_0(p)\mu^*}{2\beta^2 \pi} \dfrac{\partial}{\partial x_2} \left\{ K_0(rp\tau) - \dfrac{\beta^2}{p^2\sigma^2} \left(\dfrac{\partial^2}{\partial x_2^2} - \dfrac{\partial^2}{\partial x_1^2} \right) [K_0(rp\tau) - K_0(rp\sigma)] \right\}.
\end{cases}
$$

The Eqs. (2.6) and (2.7) should be subjected to the inverse Laplace transformation, bearing in mind the rheologic properties of the body. Let us consider the Biot model of a visco-elastic body. Assuming the same relaxation time ε^{-1} for the functions $a(t)$, $b(t)$, we have $a(t) = \mu_0 e^{-\varepsilon t}$, $b(t) = \lambda_0 c^{-\varepsilon t}$ therefore:

$$
(2.8)\qquad \lambda^* = \lambda_0 \frac{p}{p+\varepsilon}, \qquad \mu^* = \mu_0 \frac{p}{p+\varepsilon}.
$$

Thus:

$$
(2.9)\quad \beta^2 = \frac{\lambda_0 + 2\mu_0}{\mu_0} = \beta_0^2 = \text{const}, \quad \eta = \frac{p+\varepsilon}{\mu_0 p}, \quad \sigma^2 = \sigma_0^2 \frac{p+\varepsilon}{p}, \quad \tau^2 = \tau_0^2 \frac{p+\varepsilon}{p},
$$

where

$$
\sigma_0^2 = \varrho/\mu_0 = c_2^{-2}, \qquad \tau_0^2 = \frac{\varrho}{2\mu_0 + \lambda_0} = c_1^{-2}.
$$

By c_1 and c_2 we denote the velocities of propagation of the longitudinal and the transversal wave in a perfectly elastic body. Let us assume that in the direction of the x_1-axis an instantaneous unit force acts. Then $F_0(p) = 1$.

Bearing in mind that

$$
(2.10)\qquad \begin{cases}
\mathcal{L}^{-1} K_0 [r\tau_0 \sqrt{p(p+\varepsilon)}] = \gamma(r, t; \tau_0), \\[1ex]
\mathcal{L}^{-1} \left\{ \dfrac{1}{p(p+\varepsilon)} K_0 [r\tau_0 \sqrt{p(p+\varepsilon)}] \right\} = \dfrac{1}{\varepsilon} \varphi(r, t; \tau_0),
\end{cases}
$$

where

$$
(2.11)\qquad \begin{cases}
\gamma(r, t; \tau_0) = \dfrac{1}{2} \sqrt{\pi\varepsilon}\, e^{-\frac{\varepsilon t}{2}} I_{-1/2} \left(\dfrac{\varepsilon}{2} \sqrt{t^2 - r^2\tau_0^2} \right) H(t - r\tau_0), \\[2ex]
I_{-1/2}(z) = \sqrt{\dfrac{2}{\pi z}} \operatorname{ch} z, \\[2ex]
\varphi(r, t; \tau_0) = \displaystyle\int_0^t [1 - e^{-\varepsilon(t-t')}] \gamma(r, t'; \tau_0)\, dt'
\end{cases}
$$

and $H(t)$ is Heaviside's function, we obtain the following equations for stresses and strains:

$$(2.12)\quad \begin{cases} u_1(x_1, x_2, t) = \dfrac{1}{2\pi c_1^2 \varrho}\Big\{\gamma(r, t; \tau_0) + \varepsilon \displaystyle\int_0^t \gamma(r, t'; \tau_0)\, dt' - \\ \qquad - c_1^2 \dfrac{\partial^2}{\partial x_2^2}\Big[\displaystyle\int_0^t (t-t')\,[\gamma(r, t'; \tau_0) - \gamma(r, t'; \sigma_0)]\, dt'\Big]\Big\}, \\ u_2(x_1, x_2, t) = \dfrac{1}{2\pi\varrho}\, \dfrac{\partial^2}{\partial x_1\, \partial x_2} \displaystyle\int_0^t (t-t')\,[\gamma(r, t'; \tau_0) - \gamma(r, t'; \sigma_0)]\, dt'. \end{cases}$$

Next,

$$(2.13)\quad \begin{cases} \sigma_{11}^* + \sigma_{22}^* = \dfrac{\lambda_0 + \mu_0}{\varrho\pi c_1^2}\, \dfrac{\partial}{\partial x_1} \gamma(r, t; \tau_0), \\ \sigma_{11}^* - \sigma_{22}^* = \dfrac{\mu_0}{\varrho\pi\, c_1^2}\, \dfrac{\partial}{\partial x_1}\Big\{\gamma(r, t; \tau_0) - \dfrac{2c_1^2}{\varepsilon}\, \dfrac{\partial^2}{\partial x_2^2}\,[\varphi(r, t; \tau_0) - \varphi(r, t; \sigma_0)]\Big\}, \\ \sigma_{12}^* = \dfrac{\mu_0}{2\pi\varrho\, c_1^2}\, \dfrac{\partial}{\partial x_2}\Big\{\gamma(r, t; \tau_0) - c_1^2\Big(\dfrac{\partial^2}{\partial x_2^2} - \dfrac{\partial^2}{\partial x_1^2}\Big)[\varphi(r, t; \tau_0) - \varphi(r, t; \sigma_0)]\Big\}. \end{cases}$$

Knowing the displacements and the stresses for the instantaneous unit force, we can determine the displacements and stresses for a force $F = F_0(t)\, \delta(x_1)\, \delta(x_2)$ varying in time in an arbitrary manner. Thus, for instance, for the sum of the stresses we obtain:

$$(2.14)\qquad \sigma_{11}^* + \sigma_{22}^* = \frac{\lambda_0 + \mu_0}{\varrho\,\pi c_1^2}\, \frac{\partial}{\partial x_1} \int_0^t F(t')\, \gamma(r, t'; \tau_0)\, dt'.$$

The displacements and the stresses are particularly simple for the force $F_0(t) = F_0\, e^{i\omega t}$, that is for a force varying in time in a harmonic manner. In this particular case, p should be replaced by $i\omega$ in the Eqs. (2.6), (2.7) and (2.9). Thus, for instance:

$$(2.15)\qquad \sigma_{11}^* + \sigma_{22}^* = \frac{\lambda_0 + \mu_0}{\varrho\pi c_1^2}\, \frac{\partial}{\partial x_1}\, K_0\,[r\tau_0 \sqrt{i\omega(\varepsilon + i\omega)}].$$

Let us denote by $u_i^{*(1)}$ $(i = 1, 2)$ the displacement due to the action of a force directed along the x_1-axis, and by $u_i^{*(2)}$ the displacement due to a force directed along the x_2-axis.

Let us differentiate the displacements $u_i^{*(1)}$ with respect to x_1. The quantities

$$(2.16)\qquad U_i^{*(1)} = \frac{\partial u_i^{*(1)}}{\partial x_1} \qquad (i = 1, 2),$$

may be treated as displacements in the x_1 and x_2-direction due to the action of a double force whose direction coincides with the x_1-axis.

Similarly, the quantity

$$U_i^{*(2)} = \frac{\partial u_i^{*(2)}}{\partial x_2} \qquad (i=1,2), \tag{2.17}$$

is treated as a displacement in the direction of the x_i-axis due to the action of a double force acting in the x_2-direction.

Thus the quantities

$$_cU_i^* = \sum_{k=1}^{2} \frac{\partial u_i^{*(k)}}{\partial x_k} \qquad (i=1,2) \tag{2.18}$$

are the displacements due to the action of the centre of pressure located at the origin. It can easily be verified that

$$\begin{cases} _cU_1^* = -\dfrac{\eta}{\beta^2}\displaystyle\int_0^\infty\int_0^\infty \frac{\bar{F}\alpha_1 \sin\alpha_1 x_1 \cos\alpha_2 x_2}{\alpha_1^2+\alpha_2^2+p^2\tau^2}\,d\alpha_1\,d\alpha_2, \\ _cU_2^* = -\dfrac{\eta}{\beta^2}\displaystyle\int_0^\infty\int_0^\infty \frac{\bar{F}\alpha_2 \cos\alpha_1 x_1 \sin\alpha_2 x_2}{\alpha_1^2+\alpha_2^2+p^2\tau^2}\,d\alpha_1\,d\alpha_2, \end{cases} \tag{2.19}$$

or

$$_cU_i^* = \frac{\eta\bar{F}\pi}{2\beta^2}\,\frac{\partial}{\partial x_1}K_0(rp\tau) \qquad (i=1,2). \tag{2.20}$$

For the **Biot** model of a visco-elastic body, and assuming that the unit centre of pressure acts in an instantaneous manner, $(\bar{F}=1/\pi^2)$ we obtain:

$$_cU_i(x_1,x_2,t) = \frac{1}{2\pi\varrho c_1^2}\,\frac{\partial}{\partial x_i}\left[\gamma(r,t;\tau_0)+\varepsilon\int_0^t \gamma(r,t';\tau_0)\,dt'\right] \quad (i=1,2). \tag{2.21}$$

Knowing the displacements, we shall determine the stress components. Thus, for instance:

$$_c\sigma_{12}(x_1,x_2,t) = \frac{\mu_0}{\pi\varrho c_1^2}\,\frac{\partial^2}{\partial x_1\,\partial x_2}\gamma(r,t;\tau_0). \tag{2.22}$$

Let a load $F=F_0(t)\,\delta(x_1)$ act in the plane $x_1=0$. In this particular case we have $u_2=0$, $u_3=0$. The system of equations (1.10) reduces to the unique equation:

$$\beta^2 u_{1,11}^* + \eta F^* - p^2\sigma^2 u_1^* = 0. \tag{2.23}$$

Expressing the displacement and the force by an even **Fourier** integral, and bearing in mind that $\bar{F}=F_0(p)/\pi$, we obtain:

$$u_1^*(x_1,p) = \frac{F_0(p)\,\eta}{\pi\beta^2}\int_0^\infty \frac{\cos\alpha_1 x_1\,d\alpha_1}{\alpha_1^2+p^2\tau^2} = \frac{F_0(p)\,\eta}{2\beta^2 p\tau}\,e^{-p\tau x_1}. \tag{2.24}$$

For the Biot model of a visco-elastic body, and in the case of an instantaneous force, we obtain:

$$u_1^*(x_1,p)=\frac{1}{2\varrho c_1}\left(1+\frac{\varepsilon}{p}\right)\frac{e^{-x_1\tau_0\sqrt{p(p+\varepsilon)}}}{\sqrt{p(p+\varepsilon)}}. \tag{2.25}$$

Performing the inverse Laplace transformation we obtain:

$$u_1(x_1,t)=\frac{1}{2\varrho c_1}\left[\psi\left(\frac{x_1}{c_1},t\right)+\varepsilon\int_0^t\psi\left(\frac{x_1}{c_1},t'\right)dt'\right], \tag{2.26}$$

where

$$\psi\left(\frac{x_1}{c_1},t\right)=e^{-\frac{\varepsilon t}{2}}I_0\left[\frac{\varepsilon}{2}\sqrt{t^2-\left(\frac{x_1}{c_1}\right)^2}\,\right]H\left(t-\frac{x_1}{c_1}\right). \tag{2.27}$$

Since

$$\sigma_{11}^*=(\lambda^*+2\mu^*)\frac{\partial u_1^*}{\partial x_1},\qquad \sigma_{22}^*=\sigma_{33}^*=\lambda^*\frac{\partial u_1^*}{\partial x_1}, \tag{2.28}$$

therefore:

$$\left\{\begin{aligned}&\sigma_{11}(x_1,t)=\frac{\lambda_0+2\mu_0}{2\varrho c_1}\frac{\partial}{\partial x_1}\psi\left(\frac{x_1}{c_1},t\right),\\&\sigma_{22}(x_1,t)=\sigma_{33}(x_1,t)=\frac{\lambda_0}{\lambda_0+2\mu_0}\sigma_{11}(x_1,t).\end{aligned}\right. \tag{2.29}$$

For the force $F=F_0(t)\,\delta(x_1)$ we obtain:

$$\sigma_{11}(x_1,t)=\frac{\lambda_0+2\mu_0}{2\varrho c_1}\frac{\partial}{\partial x_1}\int_0^t F(t')\,\psi\left(\frac{x_1}{c_1},t'\right)dt'. \tag{2.30}$$

For the force $F=F_0\,e^{i\omega t}\,\delta(x_1)$ we obtain:

$$\sigma_{11}(x_1,t)=-\frac{\lambda_0+2\mu_0}{2\varrho c_1^2}\,e^{i\omega t}\,e^{-x_1\tau_0\sqrt{i\omega(\varepsilon+i\omega)}}. \tag{2.31}$$

3. The Axially Symmetric Problem

Let a concentrated force $F=F_0(t)\,\delta(z)\,[\delta(r)/2\pi r]$ directed along the z-axis act at the origin. Substituting the stress-strain relations

$$\left\{\begin{aligned}&(\sigma_{rr}^*,\sigma_{\varphi\varphi}^*,\sigma_{zz}^*)=\lambda^*\left(\frac{\partial u_r^*}{\partial r}+\frac{u_r^*}{r}+\frac{\partial w^*}{\partial z}\right)+2\mu^*\left(\frac{\partial u_r^*}{\partial r},\frac{u_r^*}{r},\frac{\partial w^*}{\partial z}\right),\\&\sigma_{rz}^*=\mu^*\left(\frac{\partial u_r^*}{\partial z}+\frac{\partial w^*}{\partial r}\right),\end{aligned}\right. \tag{3.1}$$

in the equations of equilibrium

$$(3.2)\qquad \begin{cases} \dfrac{\partial \sigma_r^*}{\partial r} + \dfrac{\partial \sigma_{rz}^*}{\partial z} + \dfrac{\sigma_r^* - \sigma_\varphi^*}{r} - \varrho p^2 u_r^* = 0, \\ \dfrac{\partial \sigma_{rz}^*}{\partial r} + \dfrac{\partial \sigma_{zz}^*}{\partial z} + \dfrac{\sigma_{rz}^*}{r} + F^* - \varrho p^2 w^* = 0, \end{cases}$$

we obtain the following system of displacement equations:

$$(3.3)\qquad \begin{cases} \beta^2 \left(\dfrac{\partial u_r^*}{\partial r^2} + \dfrac{1}{r} \dfrac{\partial u_r^*}{\partial r} - \dfrac{u_r^*}{r^2} \right) + \delta^2 \dfrac{\partial^2 w^*}{\partial r\, \partial z} + \dfrac{\partial^2 u_r^*}{\partial z^2} - p^2 \sigma^2 u_r^* = 0, \\ \dfrac{\partial^2 w^*}{\partial r^2} + \dfrac{1}{r} \dfrac{\partial w^*}{\partial r} + \delta^2 \dfrac{\partial}{\partial z} \left(\dfrac{\partial u_r^*}{\partial r} + \dfrac{u^*}{r} \right) + \beta^2 \dfrac{\partial^2 w^*}{\partial z^2} + \eta F^* - p^2 \sigma^2 w^* = 0. \end{cases}$$

To solve (3.3) we introduce the double Fourier-Hankel integral defined by the equations:

$$(3.4)\qquad \begin{cases} u^*(r, z, p) = \displaystyle\int_0^\infty \int_0^\infty \bar{u}_r(a, \gamma, p) \sin \gamma z\, J_1(ar)\, da\, d\gamma, \\ w^*(r, z, p) = \displaystyle\int_0^\infty \int_0^\infty \overline{w}(a, \gamma, p) \cos \gamma z\, J_0(ar)\, da\, d\gamma, \\ F^*(r, z, p) = \displaystyle\int_0^\infty \int_0^\infty \overline{F}(a, \gamma, p) \cos \gamma z\, J_0(ar)\, da\, d\gamma, \\ \dfrac{\delta(r)}{2\pi r} = \dfrac{1}{2\pi} \displaystyle\int_0^\infty a J_0(ar)\, da. \end{cases}$$

The assumption of the transformation (3.4) satisfies the symmetry condition of the functions w^* and F^* in relation to the $z = 0$ plane, and the antisymmetry condition of the displacement u_r^* in relation to that plane. Substituting the integrals (3.4) in the Eqs. (3.3), we obtain the following system of equations:

$$(3.5)\qquad \begin{cases} (\beta^2 a^2 + \gamma^2 + p^2 \sigma^2) \bar{u}_r - a\gamma\delta^2 \overline{w} = 0, \\ -a\gamma\delta^2 \bar{u}_r + (\beta^2 \gamma^2 + a^2 + p^2 \sigma^2) \overline{w} = \eta \overline{F}, \quad \overline{F} = \dfrac{1}{2\pi^2} F_0(p), \end{cases}$$

whence:

$$(3.6)\qquad \begin{cases} \bar{u}_r = \dfrac{F_0(p)\, \delta^2 \eta a}{2\pi^2 \beta^2} \, \dfrac{a\gamma}{(a^2 + \gamma^2 + p^2 \sigma^2)(a^2 + \gamma^2 + p^2 \tau^2)}, \quad \tau = \dfrac{\sigma}{\beta}, \\ \overline{w} = \dfrac{F_0(p)\, \eta a}{2\pi^2 \beta^2} \, \dfrac{\beta^2 a^2 + \gamma^2 + p^2 \sigma^2}{(a^2 + \gamma^2 + p \sigma^2)(a^2 + \gamma^2 + p^2 \tau^2)}. \end{cases}$$

Bearing in mind that

$$(3.7)\quad \int_0^\infty\int_0^\infty \frac{\alpha J_0(\alpha r)\cos\gamma z}{\alpha^2+\gamma^2+p^2\sigma^2}\,d\alpha\,d\gamma = \frac{\pi}{2} I(R;p,\sigma),\quad I=\frac{1}{R}e^{-Rp\sigma},\quad R=(z^2+r^2)^{1/2},$$

and calculating the integrals (2.4), we obtain:

$$(3.8)\qquad u_r^* = \frac{F_0(p)\eta}{4\pi p^2\sigma^2}\frac{\partial^2}{\partial r\,\partial z}\left[I(R;p,\tau)-I(R;p,\sigma)\right],$$

$$(3.9)\qquad w^* = \frac{F_0(p)\eta}{4\pi}\left\{I(R;p,\sigma)+\frac{1}{p^2\sigma^2}\frac{\partial^2}{\partial z^2}\left[I(R;p,\tau)-I(R;p,\sigma)\right]\right\}.$$

For an instantaneous point force $[F_0(p)=1]$, and for the Biot model of a visco-elastic body we obtain:

$$(3.10)\qquad u_r(r,z,t) = \frac{1}{4\pi\varrho}\frac{\partial^2}{\partial r\,\partial z}\left\{\frac{1}{R}\int_0^t\left[\Phi(R,t';\tau_0)-\Phi(R,t';\sigma_0)\right]dt'\right\},$$

$$(3.11)\qquad w(r,z,t) = \frac{1}{4\mu_0\pi}\left\{\chi(R,t)+\varepsilon\Phi(R,t;\sigma_0)+ c_2^2\frac{\partial^2}{\partial z^2}\left[\frac{1}{R}\int_0^t\left[\Phi(R,t';\tau_0)-\Phi(R,t';\sigma_0)\right]dt'\right]\right\},$$

where the following functions have been introduced

$$(3.12)\quad \Phi(R,t;\sigma_0) = \frac{1}{R}\left[e^{-\frac{\varepsilon R\sigma_0}{2}}+\frac{\varepsilon R\sigma_0}{2}\int_{R\sigma_0}^t \frac{e^{-\frac{\varepsilon v}{2}} I_1\left(\frac{\varepsilon}{2}\sqrt{v^2-R^2\sigma_0^2}\right)}{\sqrt{v^2-R^2\sigma_0^2}}\,dv\right]H(t-R\sigma_0),$$

$$(3.13)\qquad \chi(R,t) = -\frac{1}{R}\frac{\partial}{\partial(R\sigma_0)}\left[e^{-\frac{\varepsilon t}{2}} I_0\left(\frac{\varepsilon}{2}\sqrt{t^2-R^2\sigma_0^2}\right)H(t-R\sigma_0)\right] =$$

$$= \frac{1}{R}e^{-\frac{\varepsilon t}{2}}\left[\delta(t-R\sigma_0)+R\sigma_0\left(\frac{\varepsilon}{2}\right)^2\frac{I_1\left(\frac{\varepsilon}{2}\sqrt{t^2-R^2\sigma_0^2}\right)}{\frac{\varepsilon}{2}\sqrt{t^2-R^2\sigma_0^2}}H(t-R\sigma_0)\right].$$

Knowing the transforms of the displacements (3.8), (3.9), we can determine the stresses from the equations:

$$(3.14)\qquad \begin{cases} \sigma_{rr}^* = \dfrac{\lambda^*}{3\lambda^*+2\mu^*}\Lambda^*+2\mu^*\dfrac{\partial u_r^*}{\partial r}, \\[2mm] \sigma_{\varphi\varphi}^* = \dfrac{\lambda^*}{3\lambda^*+2\mu^*}\Lambda^*+2\mu^*\dfrac{u_r^*}{r}, \\[2mm] \sigma_{zz}^* = \dfrac{\lambda^*}{3\lambda^*+2\mu^*}\Lambda^*+2\mu^*\dfrac{\partial w^*}{\partial z}, \\[2mm] \sigma_{rz}^* = \mu^*\left(\dfrac{\partial u_r^*}{\partial z}+\dfrac{\partial w^*}{\partial r}\right), \end{cases}$$

where

$$\Lambda^* = \sigma^*_{rr} + \sigma^*_{\varphi\varphi} + \sigma^*_{zz} = (3\lambda^* + 2\mu^*)\left(\frac{\partial u^*_r}{\partial r} + \frac{u^*_r}{r} + \frac{\partial w^*}{\partial z}\right).$$

It can easily be verified that:

$$\Lambda^* = \frac{F_0(p)\,\eta}{4\pi\beta^2}(3\lambda^* + 2\mu^*)\frac{\partial}{\partial z} I(R, p; \tau). \tag{3.15}$$

For an instantaneous point force and for the Biot model of a visco-elastic body, we have:

$$\sigma_{rr}(r, z, t) = \frac{\lambda_0}{4\pi\mu_0\beta_0^2}\frac{\partial}{\partial z}\chi(R, t) + \frac{c_2^2}{2\pi\varepsilon}\frac{\partial^3}{\partial r^2\,\partial z}\Big\{\Phi(R, t; \tau_0) - \Phi(R, t; \sigma_0) - \int_0^t e^{-\varepsilon(t-t')}\frac{\partial}{\partial t'}[\Phi(R, t'; \tau_0) - \Phi(R, t'; \sigma_0)]\,dt'\Big\}, \tag{3.16}$$

$$\sigma_{\varphi\varphi}(r, z, t) = \frac{\lambda_0}{4\pi\mu_0\beta_0^2}\frac{\partial}{\partial z}\chi(R, t) + \frac{c_2^2}{2\pi\varepsilon r}\frac{\partial^2}{\partial r\,\partial z}\Big\{\Phi(R, t; \tau_0) - \Phi(R, t; \sigma_0) - \int_0^t e^{-\varepsilon(t-t')}\frac{\partial}{\partial t'}[\Phi(R, t'; \tau_0) - \Phi(R, t'; \sigma_0)]\,dt'\Big\}, \tag{3.17}$$

$$\sigma_{zz}(r, z, t) = \frac{\lambda_0}{4\pi\mu_0\beta_0^2}\frac{\partial}{\partial z}\chi(R, t) + \frac{1}{2\pi}\frac{\partial}{\partial z}\Big\{\chi(R, t) + \frac{c_2^2}{\varepsilon}\frac{\partial^2}{\partial z^2}\Big[\Phi(R, t; \tau_0) - \Phi(R, t; \sigma_0) - \int_0^t e^{-\varepsilon(t-t')}\frac{\partial}{\partial t'}[\Phi(R, t'; \tau_0) - \Phi(R, t'; \sigma_0)\,dt']\Big]\Big\}. \tag{3.18}$$

$$\sigma_{rz}(r, z, t) = \frac{1}{4\pi}\frac{\partial}{\partial r}\chi(R, t) + \frac{c_2^2}{2\pi\varepsilon}\frac{\partial^3}{\partial r\,\partial z^2}\Big\{\Phi(R, t; \tau_0) - \Phi(R, t; \sigma_0) - \int_0^t e^{-\varepsilon(t-t')}\frac{\partial}{\partial t'}[\Phi(R, t'; \tau_0) - \Phi(R, t'; \sigma_0]\,dt'\Big\}. \tag{3.19}$$

4. The Three-Dimensional Problem

Let a point force $F = F_0(t)\,\delta(x_1)\,\delta(x_2)\,\delta(x_3)$ act in a space, directed in the x_1-direction. Substituting in the system of equations (1.10) the integrals

$$\begin{cases} u_1^*(x_1, x_2, x_3, p) = \\ \quad = \int_0^\infty\int_0^\infty\int_0^\infty \bar{u}_1(\alpha_1, \alpha_2, \alpha_3, p)\cos\alpha_1 x_1 \cos\alpha_2 x_2 \cos\alpha_3 x_3\, d\alpha_1\, d\alpha_2\, d\alpha_3, \\ u_2^*(x_1, x_2, x_3, p) = \\ \quad = \int_0^\infty\int_0^\infty\int_0^\infty \bar{u}_2(\alpha_1, \alpha_2, \alpha_3, p)\sin\alpha_1 x_1 \sin\alpha_2 x_2 \cos\alpha_3 x_3\, d\alpha_1\, d\alpha_2\, d\alpha_3, \end{cases} \tag{4.1}$$

$$u_3^*(x_1, x_2, x_3, p) = \int_0^\infty\int_0^\infty\int_0^\infty \bar{u}_3(\alpha_1, \alpha_2, \alpha_3, p) \sin \alpha_1 x_1 \cos \alpha_2 x_2 \sin \alpha_3 x_3 \, d\alpha_1 \, d\alpha_2 \, d\alpha_3$$

we obtain the following system of equations:

$$(4.2) \quad \begin{cases} (\beta^2 \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + p^2 \sigma^2)\bar{u}_1 - \alpha_1 \alpha_2 \delta^2 \bar{u}_2 - \alpha_1 \alpha_3 \delta^2 \bar{u}_3 = \eta \bar{F}, \\ -\alpha_1 \alpha_2 \delta^2 \bar{u}_1 + (\alpha_1^2 + \beta^2 \alpha_2^2 + \alpha_3^2 + p^2 \sigma^2)\bar{u}_2 - \alpha_2 \alpha_3 \delta^2 \bar{u}_3 = 0, \\ -\alpha_1 \alpha_3 \delta^2 \bar{u}_1 - \alpha_2 \alpha_3 \delta^2 \bar{u}_2 + (\alpha_1^2 + \alpha_2^2 + \beta^2 \alpha_3^2 + p^2 \sigma^2)\bar{u}_3 = 0. \end{cases}$$

Solving for $\bar{u}_i$ $(i = 1, 2, 3)$, we obtain:

$$(4.3) \quad \begin{cases} u_1 = \dfrac{\eta \bar{F}}{\beta^2} \dfrac{(\alpha_1^2 + \alpha_2^2 + \alpha_3^2)\beta^2 - \alpha_1^2 \delta^2 + p^2 \sigma^2}{(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + p^2 \sigma)(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + p^2 \tau^2)}, \qquad \tau = \dfrac{\sigma}{\beta}, \\ \bar{u}_2 = \dfrac{\eta \bar{F} \delta^2}{\beta^2} \dfrac{\alpha_1 \alpha_2}{(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + p^2 \sigma^2)(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + p^2 \tau^2)}, \\ \bar{u}_3 = \dfrac{\eta \bar{F} \delta^2}{\beta^2} \dfrac{\alpha_1 \alpha_3}{(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + p^2 \sigma^2)(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + p^2 \tau^2)}, \qquad \bar{F} = F_0(p) \dfrac{1}{\pi^3}. \end{cases}$$

Bearing in mind that

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{\cos \alpha_1 x_1 \cos \alpha_2 x_2 \cos \alpha_3 x_3}{\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + p^2 \sigma^2} d\alpha_1 \, d\alpha_2 \, d\alpha_3 = \frac{\pi^2}{4} I(R; p, \sigma) = \frac{\pi^2}{4} \frac{1}{R} e^{-Rp\sigma},$$

we can express the displacements u_i^* $(i = 1, 2, 3)$ by the equations:

$$(4.4) \quad \begin{cases} u_1^* = \dfrac{\eta F_0(p)}{4\pi} \left\{ I(R; p, \sigma) + \dfrac{1}{p^2 \sigma^2} \dfrac{\partial^2}{\partial x_1^2} [I(R; p, \tau) - I(R; p, \sigma)] \right\}, \\ u_2^* = \dfrac{\eta F_0(p)}{4\pi p^2 \sigma^2} \dfrac{\partial^2}{\partial x_1 \partial x_2} [I(R; p, \tau) - I(R; p, \sigma)], \\ u_3^* = \dfrac{\eta F_0(p)}{4\pi p^2 \sigma^2} \dfrac{\partial^2}{\partial x_1 \partial x_3} [I(R; p, \tau) - I(R; p, \sigma]. \end{cases}$$

It is seen that the equation for u_1^* is identical with the equation for w^* of the preceding paragraph. Let us differentiate the Eqs. (4.4) with respect to x_1. Then,

$$(4.5) \qquad U_i^* = \frac{\partial u_i^*}{\partial x_1} \qquad (i = 1, 2, 3)$$

express the displacement due to the action of a double force acting at the origin and directed along the x_1-axis.

Let us differentiate the Eqs. (4.4) with respect to x_2. Then,

$$(4.6) \qquad U_i^* = \frac{\partial u_i^*}{\partial x_2},$$

express the displacement due to the action of two instantaneous forces whose vector is directed along the x_2-axis.

Let a centre of pressure act at the origin. It is identical with a system of three double forces, of which the first acts in the direction of the x_1-axis, the second acts in the direction of the x_2-axis, and the third — in the direction of the x_3-axis.

Let us denote by $u_i^{*(k)}$ $(i=1,2,3)$ the Laplace transform of the displacement in the direction of the x_i-axis due to a point force acting in the direction of the x_k-axis. Denoting by ${}_cU_i^*$ $(i=1,2,3)$ the displacement components due to the action of the centre of pressure, we obtain:

$$
{}_cU_i^* = \sum_{j=1}^{3} \frac{\partial u_i^{*(j)}}{\partial x_j}. \tag{4.7}
$$

It can easily be found that:

$$
{}_cU_i^* = \frac{\eta F_0(p)}{4\pi\beta^2}\frac{\partial}{\partial x_i} I(R; p, \tau). \tag{4.8}
$$

For the radial displacement, we obtain:

$$
{}_cU_R^* = \frac{\eta F_0(p)}{4\pi\beta^2}\frac{\partial}{\partial R} I(R; p, \tau). \tag{4.9}
$$

For the Biot model of a visco-elastic body and assuming that the action of the compression centre is continuous in time, we obtain:

$$
{}_cU_R(R,t) = \frac{1}{4\pi\varrho c_1^2}\frac{\partial}{\partial R}\left[\Phi(R,t;\tau_0) + \varepsilon\int_0^t \Phi(R,t';\tau_0)\,dt'\right], \tag{4.10}
$$

where $\Phi(R,t;\tau_0)$ is given by the Eq. (3.9).

The stresses will be obtained from the equations:

$$
\begin{cases}
\sigma_{RR} + \sigma_{\varphi\varphi} + \sigma_{\vartheta\vartheta} = \dfrac{3\lambda_0 + 2\mu_0}{4\pi\beta_0^2\mu_0}\left(\dfrac{\partial^2}{\partial R^2} + \dfrac{2}{R}\dfrac{\partial}{\partial R}\right)\Phi(R,t;\tau_0), \\[2ex]
\sigma_{\varphi\varphi} = \sigma_{\vartheta\vartheta} = \dfrac{\lambda_0}{4\pi\mu_0\beta_0^2}\left(\dfrac{\partial^2}{\partial R^2} + \dfrac{2}{R}\dfrac{\partial}{\partial R} + \dfrac{2\mu_0}{\lambda_0}\dfrac{1}{R}\dfrac{\partial}{\partial R}\right)\Phi(R,t;\tau_0).
\end{cases} \tag{4.11}
$$

If the centre of pressure varies according to the function $F(t)$, we have for instance:

$$
\sigma_{RR} + \sigma_{\varphi\varphi} + \sigma_{\vartheta\vartheta} = \frac{3\lambda_0 + 2\mu_0}{4\pi\beta_0^2\mu_0}\left(\frac{\partial^2}{\partial R^2} + \frac{2}{R}\frac{\partial}{\partial R}\right)\int_0^t \Phi(R,t';\tau_0)F(t-t')\,dt'. \tag{4.12}
$$

If $F(t) = F_0 e^{i\omega t}$, we have

$$
{}_cU_R(R,t) = \frac{F_0 e^{i\omega t}}{4\pi\beta_0^2\mu_0}\frac{\partial}{\partial R}\left[\frac{1}{R}e^{-R\tau_0\sqrt{i\omega(\varepsilon+i\omega)}}\right]\frac{i\omega+\varepsilon}{i\omega}. \tag{4.13}
$$

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Streszczenie

ROZPRZESTRZENIANIE SIĘ NAPRĘŻEŃ WYWOŁANYCH W NIEOGRANICZONEJ PRZESTRZENI LEPKO-SPRĘŻYSTEJ DZIAŁANIEM SIŁ SKUPIONYCH ZMIENIAJĄCYCH SIĘ W CZASIE

Wykonując transformację Laplace'a na równaniach przemieszczeniowych (1.4) i (1.5) ustawionych dla dwóch typów zależności między stanem naprężenia i odkształcenia (1.1) i (1.2), otrzymano układ równań przemieszczeniowych (1.10), w którym wielkości u_i^* są funkcjami miejsca i parametru transformacji p. Układ równań (1.10) rozwiązano przy użyciu całki Fouriera w sposób analogiczny do drogi postępowania obranej przez autorów pracy [1] dla ciała doskonale sprężystego. Po wyznaczeniu całek Fouriera uzyskano wielkości u_i^*, a po wykonaniu odwrotnej transformacji Laplace'a przemieszczenia u_i. W sposób szczegółowy rozpatrzono działanie siły skupionej, działanie siły równomiernie rozłożonej wzdłuż linii oraz płaszczyzny, dalej działanie skupionego i liniowego centrum ściskania. Rozpatrzono działanie siły chwilowej, ciągłej oraz w sposób harmoniczny zmieniającej się w czasie.

We wszystkich przypadkach, przy założeniu modelu Biota ciała lepko-sprężystego, uzyskano wielkości przemieszczeń i naprężeń w postaci zamkniętej.

Резюме

РАСПРОСТРАНЕНИЕ ПЕРЕМЕЩЕНИЙ В НЕОГРАНИЧЕННОМ ВЯЗКО-УПРУГОМ ПРОСТРАНСТВЕ, ВЫЗВАННОЕ ДЕЙСТВИЕМ СОСРЕДОТОЧЕННЫХ ПЕРЕМЕННЫХ ВО ВРЕМЕНИ СИЛ

Применяя трансформацию Лапласа к уравнениям в перемещениях (1.4) и (1.5), построенных для двух типов зависимости между напряженным и деформированным состоянием (1.1) и (1.2), получена система уравнений в перемещениях (1.10), в которой величины u_i^* являются функциями точки и параметра трансформаций p.

Система уравнений (1.10) решается при помощи интеграла Фурье методом аналогичным методу, примененному автором [1], в случае идеально упругого тела. После решения интегралов Фурье получены перемещения u_i^* и применяя обратное преобразование Лапласа выводятся премещения u_i. Подробно рассмотрено действие сосредоточенной силы, действие силы равномерно распределенной вдоль линий и плоскости, а также действие сосредоточенного и линейного центра давления. Рассмотрено действие силы импульсивной, гладкой и изменяющейся гармонически во времени.

Во всех случаях, принимая модель Биота вязко-упругого тела, величины перемещений и напряжений получены в замкнутом виде.

DEPARTMENT OF MECHANICS OF CONTINUOUS MEDIA
IBTP POLISH ACADEMY OF SCIENCES

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