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SOME DYNAMIC PROBLEMS OF THERMOELASTICITY

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1. Introduction

Owing to the works of M. A. Biot, [1], M. Lessen, [2], P. Chadwick and I. N. Sneddon, [3], a new trend in the research over dynamic problems of thermoelasticity has been observed since 1956, postulating a coupling between the strain field and the temperature field. In 1958, P. Chadwick and I. N. Sneddon, [3], analyzed in detail the influence of the volume and thermal changes (coupled with each other) on the form of plane harmonic waves. F. J. Lockett, [4], considered the influence of the coupled strain and temperature fields on the velocity of propagation of Rayleigh surface waves. J. N. Sneddon, [5], considered the propagation of thermal stress in thin mettalic rods [1], due to the action of periodically variable forces, and impulse, and temperature sources at the edges of the bar. In his two papers, H. Zorski, [6], [7], was concerned with stress propagation in an infinite space due to the action of a thermal impulse.

In this paper we shall be concerned with the solution of two problems. The first is that of propagation of thermal stresses in a space and semi--space, and in a space with a spherical cavity, due to the action of a point, linear and surface heat source varying in time in a harmonic manner. In solving the above problem for an elastic semi-space, the solution of a modified Lamb's problem is given, taking into account the coupling between the temperature and the strain field. The second problem concerns the stress propagation in an elastic space and semi-space due to the action of point, linear and plane strain nucleus and concentrated force.

The equations of the thermoelastic medium have the form, [1], [3]:

$$\mu \nabla^2 \mathbf{u} + (\lambda + \mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} - \varrho \frac{\partial^2 \mathbf{u}}{\partial t^2} = a_t (2\mu + 3\lambda)\,\mathrm{grad}\,\Theta, \tag{1.1}$$

$$\nabla^2 \Theta - \frac{1}{\varkappa} \frac{\partial \Theta}{\partial t} - \eta \frac{\partial}{\partial t}\,\mathrm{div}\,\mathbf{u} = -\frac{Q(P,t)}{\varkappa}, \tag{1.2}$$

[1] By the courtesy of Prof. I. N. Sneddon I was able to familiarise myself with that work before it was published.

where $\mathbf{u}$ is the displacement vector, Θ temperature (assuming that $\Theta+T$ is the absolute temperature and the state $\Theta=0$ concerns the state where the stresses and strains are zero). μ, λ are Lame's constants, ϱ denotes the density, a_t the coefficient of thermal linear dilatation, $\varkappa$ the coefficient of heat conduction. Next, $Q=W/\varrho c$, $\eta=\gamma_0 T/\varrho c\varkappa$. W is the quantity of heat produced in the body per unit time and volume, c — the specific heat and $\gamma_0=a_t(3\lambda+2\mu)$. Substituting in (1.1) and (1.2) the displacement function

$$\mathbf{u}=\operatorname{grad}\Phi+\operatorname{rot}\Psi, \tag{1.3}$$

we shall reduce the system of displacement equations to the three equations:

$$\left(\nabla^2-\frac{1}{c_1^2}\frac{\partial}{\partial t^2}\right)\Phi=\vartheta_0\Theta, \tag{1.4}$$

$$\left(\nabla^2-\frac{1}{\varkappa}\frac{\partial}{\partial t}\right)\Theta-\eta\frac{\partial}{\partial t}\nabla^2\Phi=-\frac{Q(P,t)}{\varkappa}, \tag{1.5}$$

$$\left(\nabla^2-\frac{1}{c_2^2}\frac{\partial^2}{\partial t^2}\right)\operatorname{rot}\vec{\Psi}=0. \tag{1.6}$$

In these equations, $c_1=[(\lambda+2\mu)/\varrho]^{1/2}$ is the velocity of propagation of the longitudinal elastic wave, $c_2=(\mu/\varrho)^{1/2}$ the velocity of propagation of the transversal elastic wave and $\vartheta_0=a_t(3\lambda+2\mu)/(\lambda+2\mu)$.

2. The Stresses Due to the Action of Heat Sources in an Infinite Space

Let us consider the propagation of thermoelastic waves in an infinite space, and study in turn the action of a point, linear and plane source of heat. Let us assume that these heat sources vary in time in a periodic manner, therefore $Q(P,t)=e^{i\omega t}Q_0(P)$, where ω is a positive real number and denotes the frequency of heat source. In the case of periodic variation of the heat source we have:

$$\Theta(P,t)=e^{i\omega t}\Theta^*(P), \quad \Phi(P,t)=e^{i\omega t}\Phi^*(P), \quad \Psi(P,t)=e^{i\omega t}\Psi^*(P). \tag{2.1}$$

Substituting these functions in the Eqs. (1.4)—(1.6) we obtain:

$$(\nabla^2+\sigma^2)\Phi^*=\vartheta_0\Theta^*, \tag{2.2}$$

$$(\nabla^2-q)\Theta^*-q\eta'\nabla^2\Phi=-\frac{Q_0(P)}{\varkappa}, \tag{2.3}$$

$$(\nabla^2+\tau^2)\operatorname{rot}\Psi^*=0, \tag{2.4}$$

where

$$\sigma^2=\frac{\omega^2}{c_1^2}, \quad \tau^2=\frac{\omega^2}{c_2^2}, \quad q=\frac{i\omega}{\varkappa}, \quad \eta'=\eta\varkappa.$$

Eliminating successively from (2.2) and (2.3) the function Φ^* and the function Θ^* we obtain.

$$(2.5)\qquad (\nabla^2 - q)(\nabla^2 + \sigma^2)\Theta^* - q\varepsilon\nabla^2\Theta^* = -\frac{1}{\varkappa}(\nabla^2 + \sigma^2)\Theta_0(P),$$

$$(2.6)\qquad (\nabla^2 - q)(\nabla^2 + \sigma^2)\Phi^* - q\varepsilon\nabla^2\Phi^* = -\frac{\vartheta_0}{\varkappa}Q_0(P), \quad \varepsilon = \eta\varkappa\vartheta_0.$$

It suffices to solve the Eq. (2.6) because the knowledge of Φ^* suffices to determine from the equation (2.2) the function Θ^*.

Let a point heat source $Q(P,t) = e^{i\omega t}Q_0\delta(R)$ act at the origin. The most convenient way is to solve (2.6) in cylindrical coordinates. This equation will take the form

$$(2.7)\qquad \begin{cases} (\nabla^2 - q)(\nabla^2 + \sigma^2)\Phi^* - q\varepsilon\nabla^2\Phi^* = -\dfrac{\vartheta_0 Q_0}{2\pi r}\delta(r)\delta(z), \\ \nabla^2 = \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{\partial^2}{\partial z^2}, \\ \dfrac{\delta(r)}{2\pi r}\delta(z) = \dfrac{1}{2\pi^2}\displaystyle\int_0^\infty\int_0^\infty aJ_0(ar)\cos\gamma z\, da\, d\gamma. \end{cases}$$

Expressing the function Φ^* by means of the Fourier-Hankel integral

$$(2.8)\qquad \Phi^*(r,z) = \int_0^\infty\int_0^\infty A(a,\gamma)J_0(ar)\cos\gamma z\, da\, d\gamma,$$

we can represent the solution of the Eq. (2.6) in the form:

$$(2.9)\qquad \Phi^* = -\frac{Q_0\vartheta_0}{2\pi^2\varkappa}\int_0^\infty\int_0^\infty \frac{aJ_0(ar)}{F(a,\gamma)}\cos\gamma z\, da\, d\gamma,$$

where

$$F(a,\gamma) = (a^2+\gamma^2)^2 + [q(1+\varepsilon) - \sigma^2](a^2+\gamma^2) - q\sigma^2 = (a^2+\gamma^2+k_1^2)(a^2+\gamma^2+k_2^2),$$

and

$$k_1^2 + k_2^2 = q(1+\varepsilon) - \sigma^2, \quad k_1^2 k_2^2 = -q\sigma^2.$$

After performing the integrations required, we can represent the function Φ^* in the closed form:

$$(2.10)\qquad \Phi^* = \frac{Q_0\vartheta_0}{4\pi\varkappa R(k_1^2 - k_1^2)}(e^{-k_1R} - e^{-k_2R})$$

where

$$R = (r^2+z^2)^{1/2}, \quad r = (x^2+y^2)^{1/2}, \quad k_{1,2} = a_{1,2} + ib_{1,2}, \quad a_{1,2} > 0.$$

From (2.2) we obtain:

$$\Theta^* = \frac{Q_0}{4\pi\varkappa R(k_1^2 - k_2^2)} \left[(\sigma^2 + k_1^2)\, e^{-k_1 R} - (\sigma^2 + k_2^2)\, e^{-k_2 R}\right]. \tag{2.11}$$

The knowledge of the function Φ^* enables the determination of the stresses from the Eqs. [9]

$$\sigma_{ij} = 2\mu\left(\frac{\partial^2}{\partial x_i \partial x_j} - \delta_{ij}\nabla^2\right)\Phi + \delta_{ij}\varrho\frac{\partial^2 \Phi}{\partial t^2} \quad (i,j = 1,2,3). \tag{2.12}$$

In our case of spherical symmetry, we have:

$$\begin{cases} \sigma_{RR} = -e^{i\omega t}\left(\dfrac{4\mu}{R}\dfrac{\partial \Phi^*}{\partial R} + \varrho\,\omega^2 \Phi^*\right), \\ \sigma_{\varphi\varphi} = \sigma_{\vartheta\vartheta} = -e^{i\omega t}\left[2\mu\left(\dfrac{d^2\Phi^*}{dr^2} + \dfrac{1}{R}\dfrac{d\Phi^*}{dR}\right) + \varrho\,\omega^2\Phi^*\right]. \end{cases} \tag{2.13}$$

In view of the spherical symmetry of stress and temperature field we are concerned with a modified longitudinal elastic wave, so that: $\Psi = 0$.

If the influence of the coupling between the temperature field and the strain field is disregarded (or, in other words if $\varepsilon = 0$), we have $k_1 = \sqrt{q}$, $k_2 = i\sigma$. In this case, the equations (2.10) and (2.11) become, [10],

$$\Phi^* = \frac{Q_0 \vartheta_0}{4\pi\varkappa R(\sigma^2 + q)}\left(e^{-R\sqrt{q}} - e^{-Ri\sigma}\right), \qquad \Theta^* = \frac{Q_0}{4\pi\varkappa R}\, e^{-R\sqrt{q}}. \tag{2.14}$$

Let a linear heat source act along the z-axis $Q(P,t) = e^{i\omega t} Q_0 (\delta(r)/2\pi r)$. In this case, (2.6) becomes:

$$(\nabla_r^2 - q)(\nabla_r^2 + \sigma^2)\Phi^* - q\varepsilon\nabla_r^2\Phi^* = -\frac{Q_0\vartheta_0}{2\pi\varkappa}\int_0^\infty \alpha J_0(\alpha r)\, d\alpha, \tag{2.15}$$

$$\nabla_r^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}.$$

Expressing $\Phi^*(r)$ by means of a Hankel integral

$$\Phi^*(r) = \int_0^\infty A(\alpha) J_0(\alpha r)\, d\alpha,$$

we represent the solution of the Eq. (2.15) in the form:

$$\Phi^*(r) = -\frac{Q\vartheta_0}{2\pi\varkappa}\int_0^\infty \frac{\alpha J_0(\alpha r)}{F(\alpha)}\, d\alpha, \tag{2.16}$$

where

$$F(\alpha) = \alpha^4 + [q(1+\varepsilon) - \sigma^2]\,\alpha^2 - q\varepsilon = (\alpha^2 + k_1^2)(\alpha^2 + k_2^2).$$

Performing the integration required in (2.16), we obtain:

$$\Phi^*(r) = \frac{Q_0 \vartheta_0}{2\pi\varkappa (k_1^2 - k_2^2)} \left[K_0(k_1 r) - K_0(k_2 r)\right], \tag{2.17}$$

where $K_0(k_{1,2} r)$ denotes the modified Bessel function of the third kind and zero order. From (2.2) we find that

$$\Theta^*(r) = \frac{Q_0}{2\pi\varkappa (k_1^2 - k_2^2)} \left[(\sigma^2 + k_1^2) K_0(k_1 r) - (\sigma^2 + k_2^2) K_0(k_2 r)\right]. \tag{2.18}$$

The stresses σ_{ij} will be found from the equations:

$$\begin{cases} \sigma_{rr} = -e^{i\omega t}\left(\frac{2\mu}{r}\frac{d\Phi^*}{dr} + \varrho\omega^2\Phi^*\right), \\ \sigma_{\varphi\varphi} = -e^{i\omega t}\left(2\mu\frac{d^2\Phi}{dr^2} + \varrho\omega^2\Phi^*\right). \end{cases} \tag{2.19}$$

In the particular case of $\varepsilon = 0$, we obtain [10]:

$$\Phi^* = \frac{Q_0\vartheta_0}{2\pi\varkappa(\sigma^2 + q)}\left[K_0(r\sqrt{q}) - K_0(i\sigma r)\right], \qquad \Theta^* = \frac{Q_0}{2\pi\varkappa} K_0(r\sqrt{q}). \tag{2.20}$$

Let in $x = 0$ plane, a plane source of heat $Q(P, t) = e^{i\omega t} Q_0 \delta(x)$ act. Our aim is to solve the equation:

$$\left(\frac{d^2}{dx^2} - q\right)\left(\frac{d^2}{dx^2} + \sigma^2\right)\Phi^* - q\varepsilon\frac{d^2\Phi^*}{dx^2} = -\frac{\vartheta_0 Q_0}{\varkappa}\delta(x). \tag{2.21}$$

Using the F o u r i e r transform we obtain the following equations for the functions $\Phi^*(x)$ and $\Theta^*(x)$:

$$\Phi^*(x) = \frac{Q_0\vartheta_0}{2\varkappa(k_1^2 - k_2^2)}\left(\frac{e^{-k_1 x}}{k_1} - \frac{e^{-k_2 x}}{k_2}\right), \quad x > 0 \tag{2.22}$$

$$\Theta^*(x) = \frac{Q_0}{2\varkappa(k_1^2 - k_2^2)}\left[(\sigma^2 + k_1^2)\frac{e^{-k_1 x}}{k_1} - (\sigma^2 + k_2^2)\frac{e^{-k_2 x}}{k_2}\right], \quad x > 0. \tag{2.23}$$

The stresses σ_{ij} are given by the equations:

$$\begin{cases} \sigma_{xx} = -e^{i\omega t}\varrho\omega^2\Phi^*, \\ \sigma_{yy} = \sigma_{zz} = -e^{i\omega t}\left(2\mu\frac{d^2\Phi^*}{dx^2} + \varrho\omega^2\Phi^*\right). \end{cases} \tag{2.24}$$

If we reject the influence of the coupling between the temperature field and the strain field, then, [10]:

$$\Phi^* = \frac{Q_0\vartheta_0}{2\varkappa(q + \sigma^2)}\left(\frac{e^{-x\sqrt{q}}}{\sqrt{q}} - \frac{e^{-xi\sigma}}{i\sigma}\right), \qquad \Theta^* = \frac{Q_0 e^{-x\sqrt{q}}}{2\varkappa\sqrt{q}}, \quad x > 0. \tag{2.25}$$

Let us consider now the roots k_1, k_2 of the equation:

$$(2.26)\qquad (k^2+k_1^2)(k^2+k_2^2)=0, \quad k_1^2+k_2^2=q(1+\varepsilon)-\sigma^2, \quad k_2^2k_2^2=-q\sigma^2.$$

Let us represent them in the form:

$$(2.27)\qquad k_{1,2}=\frac{c_1}{\varkappa}\left[\frac{1}{2}(i\varrho\xi-\varrho^2)\pm\Delta\right]^{1/2}, \qquad \Delta=(\varrho^2(\varrho^2-\xi^2)+2i\varrho^3(2-\xi))^{1/2},$$

$$\xi=1+\varepsilon$$

where $\varrho=\omega/\omega^*$ is a dimensionless quantity and $\omega^*=c_1^2/\varkappa$ characteristic for the thermoelastic medium introduced by I. N. Sneddon, [8].

Let us observe that

$$\Delta=e+if, \quad e=\frac{1}{\sqrt{2}}[(g^2+h^2)^{1/2}+g]^{1/2}, \quad f=\frac{1}{\sqrt{2}}[(g^2+h^2)^{1/2}-g]^{1/2},$$

where

$$g=\varrho^2(\varrho^2-\xi^2), \quad h=2\varrho^3(2-\xi).$$

Since

$$g^2+h^2=\varrho^4[(\varrho^2-\xi^2)^2+4(2-\xi^2)^2\varrho^2]>0,$$

therefore

$$f>0, \quad e>0.$$

Introducing the auxiliary values

$$\bar{a}_{1,2}=\frac{1}{\sqrt{2}}[(c_{1,2}^2+d_{1,2}^2)^{1/2}+c_{1,2}]^{1/2}, \quad \bar{b}_{1,2}=\frac{1}{\sqrt{2}}[(c_{1,2}^2+d_{1,2}^2)^{1/2}-c_{1,2}]^{1/2},$$

where

$$c_{1,2}=\frac{1}{2}(-\varrho^2\pm e), \quad d_{1,2}=\frac{1}{2}(\varrho\xi\pm f),$$

we determine the roots $k_{1,2}$ from the equations:

$$k_{1,2}=\frac{c_1}{\varkappa}(\bar{a}_{1,2}+i\bar{b}_{1,2}), \quad \bar{a}_{1,2}>0.$$

The root k_1 corresponds to a modified thermal wave, k_2 to a modified longitudinal elastic wave because for $\varepsilon=0$ we have $k_1=\sqrt{q}$, $k_2=i\sigma$. It was mentioned above that we are interested only in those roots the real part of which is positive, because only such roots concern modified thermal and elastic waves propagating from the centre to infinity. In their interesting work, P. Chadwick, and I. N. Sneddon, [3], study in detail the behaviour of the roots k_1, k_2 depending on the parameter ε, and obtain approximate values for $\varrho\ll 1$ and $\varrho\gg 1$ expanding the determinant Δ in power series in ϱ and ϱ^{-1}. It is shown, that for $\varrho \ll 1$ we have:

$$(2.28)\quad \begin{cases} \dfrac{\varkappa}{c_1} k_1 = \left(\dfrac{1}{2}(1+\varepsilon)\varrho\right)^{1/2}\left[\left(1-\dfrac{\varrho\varepsilon}{2(1+\varepsilon)^2}\right)+i\left(1+\dfrac{\varrho\varepsilon}{2(1+\varepsilon)^2}\right)\right], \\ \dfrac{\varkappa}{c_1} k_2 = \dfrac{\varrho^2\varepsilon}{2(1+\varepsilon)^{5/2}}+i\dfrac{\varrho}{(1+\varepsilon)^{1/2}}, \qquad 0<\varrho<10^{-2}. \end{cases}$$

For $\varrho \gg 1$ the above authors obtained the following approximate values:

$$\frac{\varkappa}{c_1} k_1 = \left(\frac{1}{2}\varrho\right)^{1/2}\left[\left(1-\frac{\varepsilon}{2\varrho}\right)+i\left(1+\frac{\varepsilon}{2\varrho}\right)\right], \qquad \frac{\varkappa}{c_1} k_2 = \frac{1}{2}\varepsilon + i\varrho .$$

They analyzed in detail the influence of the parameter ε on the velocity of propagation of longitudinal waves and the coefficient of attenuation. The results obtained concern the influence of the coupled temperature field on plane harmonic waves. But it is evident that these results may directly be transferred to cylindrical and spherical waves because in both cases we have the same roots k_1, k_2. For $\varrho \ll 1$ the velocity of the modified elastic wave is [2]

$$c_1 \approx \left(1+\frac{\varepsilon}{2}\right)c_1^0,$$

where c_1^0 denotes the velocity of propagation of the elastic wave for the non-coupled problem, therefore for $\varepsilon = 0$. The coefficient of attenuation has the form:

$$\chi \approx \frac{\varepsilon}{2}(2-5\varepsilon)\varrho^2 .$$

The quantity ε being small ($\varepsilon = 3.56\cdot10^{-2}$ for aluminium $\varepsilon = 2.97\cdot10^{-4}$ for steel, $\varepsilon = 7.33\cdot10^{-2}$ for lead), it is seen that the influence of coupling of the temperature field and the stress field on the velocity of propagation of plane, cylindrical and spherical waves is insignificant. Small differences are observed also in stresses in relation to those obtained for $\varepsilon = 0$.

3. Stresses Due to the Action of Heat Sources in an Elastic Semi-Space

The stress obtained or a plane source of heat acting in an infinite space enables us to study a few examples of action of a plane source of heat in an elastic semi-space.

Let us consider first the action of a plane source of heat in the $x = \xi$ plane, $\xi \gg 0$. Let us assume that in the $x = 0$ plane, bounding the elastic semi-space we have $\Theta = 0$, $\sigma_{xx} = 0$. These conditions will be satisfied using

[2] See [3].

the reflection method — consisting in locating in an infinite space two plane heat sources, one positive in the $x=\xi$ plane, the other negative in the $x=-\xi$ plane

Bearing in mind the equations (2.22), (2.23), we obtain:

$$\Phi^* = \frac{Q_0\vartheta_0}{2\varkappa(k_1^2-k_2^2)}\left[\frac{1}{k_1}\left(e^{-(x-\xi)k_1}-e^{-(x+\xi)k_1}\right)-\right.$$
$$\left.-\frac{1}{k_2}\left(e^{-(x-\xi)k_2}-e^{-(x+\xi)k_2}\right)\right],\quad x>\xi, \tag{3.1}$$

$$\Theta^* = \frac{Q_0}{2\varkappa(k_1^2-k_2^2)}\left[\frac{\sigma^2+k_1^2}{k_1}\left(e^{-(x-\xi)k_1}-e^{-(x+\xi)k_1}\right)-\right.$$
$$\left.-\frac{\sigma^2+k_2^2}{k_2}\left(e^{-(x-\xi)k_2}-e^{-(x+\xi)k_2}\right)\right],\quad x>\xi, \tag{3.2}$$

and

$$\Phi^* = \frac{Q_0\vartheta_0}{2\varkappa(k_1^2-k_2^2)}\left[\frac{1}{k_1}e^{(x-\xi)k_1}-e^{-(x+\xi)k_1})-\right.$$
$$\left.-\frac{1}{k_2}\left(e^{(x-\xi)k_2}-e^{-(x+\xi)k_2}\right)\right],\quad x<\xi, \tag{3.3}$$

$$\Theta^* = \frac{Q_0}{2\varkappa(k_1^2-k_2^2)}\left[\frac{\sigma^2+k_1^2}{k_1}\left(e^{(x-\xi)k_1}-e^{-(x+\xi)k_1}\right)-\right.$$
$$\left.-\frac{\sigma^2+k_2^2}{k_2}\left(e^{(x-\xi)k_2}-e^{-(x+\xi)k_2}\right)\right],\quad x<\xi. \tag{3.4}$$

The stresses $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$ will be found from (2.24). From the Eqs. (3.3) and (3.4), it is seen that for $x=0$ we have $\Theta^*=0$ and $\Phi^*(0)=0$. From the first of the equations (2.24) we shall see that for $\Phi^*(0)=0$ we have also $\sigma_{zz}(0,t)=0$ which was to be obtained.

Let a plane source of heat act in the plane $x=\xi$, $\xi>0$ of the elastic semi-space. Let us assume that in the plane $x=0$ we have $\sigma_{xx}=0$ and $\partial\Theta/\partial x=0$.

Let two positive heat sources act in the infinite space, one in the $x=\xi$ plane, the other in the $x=-\xi$ plane. In this way we shall satisfy one of the boundary conditions, the condition $[\partial\Theta/\partial x]_{x=0}=0$. We have:

$$\Phi^* = \frac{Q_0\vartheta_0}{2\varkappa(k_1^2-k_2^2)}\left[\frac{1}{k_1}\left(e^{-(x-\xi)k_1}+e^{-(x+\xi)k_1}\right)-\right.$$
$$\left.-\frac{1}{k_2}\left(e^{-(x-\xi)k_2}+e^{-(x+\xi)k_2}\right)\right],\quad x>\xi, \tag{3.5}$$

$$\Theta^* = \frac{Q_0}{2\varkappa(k_1^2-k_2^2)}\left[\frac{\sigma^2+k_1^2}{k_1}\left(e^{-(x-\xi)k_1}+e^{-(x+\xi)k_1}\right)-\right.$$
$$\left.-\frac{1}{k_2}\left(e^{-(x-\xi)k_2}+e^{-(x+\xi)k_2}\right)\right],\quad x>\xi, \tag{3.6}$$

and

$$(3.7)\qquad \Phi^* = \frac{Q_0 \vartheta_0}{2\varkappa (k_1^2 - k_2^2)} \left[\frac{1}{k_1} (e^{(x-\xi)k_1} - e^{-(x+\xi)k_1}) - \right.$$
$$\left. - \frac{1}{k_2} (e^{-(x-\xi)k_2} + e^{-(x+\xi)k_2}) \right], \quad x < \xi,$$

$$(3.8)\qquad \Theta^* = \frac{Q_0}{2\varkappa (k_1^2 - k_2^2)} \left[\frac{\sigma^2 + k_1^2}{k_1} (e^{(x-\xi)k_1} + e^{-(x+\xi)k_1}) - \right.$$
$$\left. - \frac{\sigma^2 + k^2}{k_2} (e^{(x-\xi)k_2} + e^{-(x+\xi)k_2}) \right], \quad x < \xi.$$

Knowing the function $\bar{\Phi}^*$, we can determine the stress $\bar{\sigma}_{11}^* = -\varrho\omega^2\bar{\Phi}^*$ in the cross-section $x = 0$. We obtain:

$$(3.9)\qquad \sigma_{xx}^*(0) = -\frac{Q\vartheta_0 \omega^2 \varrho}{\varkappa (k_1^2 - k_2^2)} \left(\frac{e^{-\xi k_1}}{k_1} - \frac{e^{-\xi k_2}}{k_2} \right).$$

This stress is different from zero. Since the function Φ^* does not satisfy the boundary condition $\bar{\sigma}_{zz}^*(0) = 0$, we should add to the state $\bar{\sigma}_{ij}$, the state $\bar{\sigma}_{ij}$ so chosen that the boundary conditions are satisfied. The stress $\bar{\bar{\sigma}}_{xx}$ will be expressed by means of the function $\bar{\bar{\Phi}}^*$ satisfying the homogeneous equation (2.6):

$$(3.10)\qquad \left(\frac{d^2}{dx^2} - q \right) \left(\frac{d^2}{dx^2} + \sigma^2 \right) \bar{\Phi}^* - q\varepsilon \frac{d^2 \Phi^*}{dx^2} = 0,$$

where

$$(3.11)\qquad \bar{\sigma}_{xx}^* = -\varrho\omega^2 \Phi^*.$$

A solution of the Eq. (3.10) is the function:

$$(3.12)\qquad \bar{\bar{\Phi}}^* = C_1 e^{-k_1 x} + C_2 e^{-k_2 x}$$

assuming that $\bar{\Phi}^* = 0$ at infinity.

From the boundary condition

$$(3.13)\qquad \bar{\sigma}_{xx}^*(0) + \bar{\bar{\sigma}}_{xx}^*(0) = 0$$

we obtain:

$$(3.14)\qquad C_1 + C_2 + \frac{Q\vartheta_0}{\varkappa (k_1^2 - k_2^2)} \left(\frac{e^{-k_1 \xi}}{k_1} - \frac{e^{-k_2 \xi}}{k_2} \right) = 0.$$

But the state of stress $\bar{\sigma}_{ij}^*$ will provoke an additional temperature field $\bar{\Theta}^*$, connected with the function $\bar{\bar{\Phi}}^*$ by the equation:

$$(3.15)\qquad \left(\frac{d^2}{dx^2} + \sigma^2 \right) \bar{\Phi}^* = \vartheta_0 \bar{\Theta}^*.$$

Hence:

$$(3.16)\qquad \vartheta_0 \bar{\bar{\Theta}}^* = C_1 (k_1^2 + \sigma^2)\, e^{-k_1 x} + C_2 (k_2^2 + \sigma^2)\, c^{-k_2 x}.$$

From the condition $d\bar{\bar{\Theta}}/dx = 0$ for $x = 0$, we obtain

$$k_1(k_1^2 + \sigma^2)C_1 + k_2(k_2^2 + \sigma^2)C_2 = 0. \tag{3.17}$$

From the equations (3.14) and (3.17), we obtain the constants C_1, C_2:

$$\begin{cases} C_1 = -\dfrac{Q\vartheta_0 k_2(k_2^2 + \sigma^2)}{\varkappa(k_1^2 - k_2^2)M}\left(\dfrac{e^{-k_1\xi}}{k_1} - \dfrac{e^{-k_2\xi}}{k_2}\right), \\ C_2 = \dfrac{Q\vartheta_0 k_1(k_1^2 + \sigma^2)}{\varkappa(k_1^2 - k_2^2)M}\left(\dfrac{e^{-k_1\xi}}{k_1} - \dfrac{e^{-k_2\xi}}{k_2}\right), \\ M = k_2(k_1^2 - \sigma^2) - k_1(k_2^2 + \sigma^2). \end{cases} \tag{3.18}$$

Final stresses will be obtained from the equations:

$$\begin{cases} \sigma_{xx} = -\varrho\omega^2 e^{i\omega t}(\bar{\Phi}^* + \bar{\bar{\Phi}}^*), \\ \sigma_{yy} = \sigma_{zz} = -e^{i\omega t}\left[2\mu\dfrac{d^2}{dx^2}(\Phi^* + \bar{\Phi}^*) + \varrho\omega^2(\bar{\Phi}^* + \bar{\bar{\Phi}}^*)\right]. \end{cases} \tag{3.19}$$

The solution of the next problem is more complicated. Let a concentrated heat source act at the point $(0, 0, \xi)$ of the elastic semi-space varying in function of time in a harmonic manner. Let the plane $z = 0$ bounding the semi-space be free from stress. Let us assume that $\Theta = 0$ in this plane.

Using the reflection method we locate at the point $(0, 0, \zeta)$ of the infinite space a positive heat source, and at the point $(0, 0, -\zeta)$ a negative heat source. In this way we shall satisfy the condition $\bar{\Theta} = 0$ and $\bar{\sigma}_{zz} = 0$ in the plane $z = 0$. On the other hand, the stress $\bar{\sigma}_{rz}$ will be different from zero in this plane. Using the Eqs. (2.10) and (2.11), we find that

$$\Phi^* = \frac{Q_0\vartheta_0}{4\pi\varkappa(k_1^2 - k_2^2)}\left[\frac{1}{R_1}(e^{-k_1R_1} - e^{-k_2R_1}) - \frac{1}{R_2}(e^{-k_1R_2} - e^{-k_2R_2})\right] \tag{3.20}$$

and

$$\Theta^* = \frac{Q_0}{4\pi\varkappa(k_1^2 - k_2^2)}\left[\frac{(\sigma^2 + k_1^2)e^{-k_1R_1} - (\sigma^2 + k_2^2)e^{-k_2R_1}}{R_1} - \frac{(\sigma^2 + k_1^2)e^{-k_1R_2} - (\sigma^2 + k_2^2)e^{-k_2R_2}}{R_2}\right], \tag{3.21}$$

where

$$R_{1,2} = [r^2 + (z \mp \zeta)^2]^{1/2}, \quad r = (x^2 + y^2)^{1/2}.$$

The stress $\bar{\sigma}^*_{rz}$ is expressed by the relation:

$$\bar{\sigma}^*_{rz} = 2\mu\frac{\partial^2\bar{\Phi}^*}{\partial r\partial z}. \tag{3.22}$$

Expressing the function $\bar{\Phi}^*$ by means of the Hankel-Fourier integral [using the Eq. (1.9)], we obtain:

$$\bar{\sigma}^*_{rz}(r,z) = \frac{2\mu Q_0 \vartheta_0}{\pi^2 \varkappa} \int_0^\infty \int_0^\infty \frac{a^2 J_1(ar)}{F(a,\gamma)} \gamma \sin \gamma\xi \cos \gamma z \, da d\gamma. \tag{3.23}$$

Hence

$$\bar{\sigma}^*_{rz}(r,0) = \frac{2Q_0 \vartheta_0 \mu}{\pi^2 \varkappa} \int_0^\infty \int_0^\infty \frac{a^2 J_1(ar)}{F(a,\gamma)} \gamma \sin \gamma\xi \, da d\gamma = \tag{3.24}$$

$$= -\frac{Q\vartheta_0 \mu}{\pi \varkappa (k_1^2 - k_2^2)} \int_0^\infty \left(e^{-\xi\sqrt{k_1^2+a^2}} - e^{-\xi\sqrt{k_2^2+a^2}}\right) a^2 J_1(ar) \, da$$

It is seen that in the plane $z=0$ we have $\bar{\sigma}^*_{rz}(r,0) \neq 0$. To render the $z=0$ plane free from stress, we should solve an additional problem (generalized Lamb's problem). We should determine in the elastic semi-space (with no heat source) the state of stress $\bar{\bar{\sigma}}_{ij}$ so chosen that in the $z=0$ plane the following boundary conditions are satisfied:

$$\bar{\sigma}^*_{rz} + \bar{\bar{\sigma}}^*_{rz} = 0, \quad \bar{\bar{\sigma}}^*_{zz} = 0, \quad \bar{\bar{\Theta}}^* = 0. \tag{3.25}$$

In an elastic semi-space the temperature field $\bar{\bar{\Theta}}^*$ will appear with the longitudinal and transversal wave. The following wave equations should be satisfied in the elastic semi-space:

$$(\nabla^2 - q)(\nabla^2 + \sigma^2)\bar{\bar{\Phi}}^* - q\varepsilon\nabla^2\Phi^* = 0, \tag{3.26}$$

$$(\nabla^2 + \tau^2) \operatorname{rot} \Psi^* = 0. \tag{3.27}$$

The temperature field $\bar{\bar{\Theta}}^*$ will be determined from the equation:

$$(\nabla^2 + \sigma^2)\bar{\bar{\Phi}}^* = \vartheta_0 \bar{\bar{\Theta}}^*.$$

The stresses $\bar{\bar{\sigma}}_{rz}$, σ_{zz} will be composed of two parts: the part connected with the function $\bar{\bar{\Phi}}$, and that connected with the function Ψ. According to the Eqs. (2.12), if we pass to the system of cylindrical coordinates we have:

$$\bar{\sigma}'_{zz} = 2\mu\left(\frac{\partial^2}{\partial z^2} - \nabla^2\right)\Phi + \varrho \frac{\partial^2 \bar{\bar{\Phi}}}{\partial t^2}, \quad \bar{\sigma}'_{rz} = 2\mu \frac{\partial^2 \bar{\bar{\Phi}}}{\partial r \partial z}. \tag{3.28}$$

The stresses connected with the function Ψ are expressed by the relations

$$\bar{\bar{\sigma}}''_{zz} = \lambda e'' + 2\mu \frac{\partial w''}{\partial z}, \quad \bar{\bar{\sigma}}''_{zr} = \mu\left(\frac{\partial u''_r}{\partial z} + \frac{\partial w''}{\partial r}\right), \tag{3.29}$$

where e'' denotes the dilatation. Bearing in mind that [3]

$$u''_r = \frac{\partial^2 \Psi}{\partial r \partial z}, \quad w'' = -\frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r}\frac{\partial \Psi}{\partial r} = \frac{\partial^2 \Psi}{\partial z^2} - \frac{1}{c_2^2}\frac{\partial^2 \Psi}{\partial t^2}, \quad e'' = 0,$$

[3] See for instance [11].

we obtain:

$$(3.30)\qquad \begin{cases} \sigma''_{zz} = 2\mu \dfrac{\partial}{\partial z}\left(\dfrac{\partial^2}{\partial z^2} - \dfrac{1}{c_2^2}\dfrac{\partial^2}{\partial t^2}\right)\Psi, \\ \sigma''_{rz} = \mu \dfrac{\partial}{\partial r}\left(2\dfrac{\partial^2}{\partial z^2} - \dfrac{1}{c_2^2}\dfrac{\partial^2}{\partial t^2}\right)\Psi. \end{cases}$$

Adding the stresses σ'_{rz}, $\overline{\sigma}''_{rz}$ and the stresses $\overline{\sigma}'_{zz}$, $\overline{\overline{\sigma}}''_{zz}$, and taking their amplitudes, we obtain:

$$(3.31)\qquad \begin{cases} \sigma^*_{zz} = 2\mu\left(\dfrac{\partial^2}{\partial z^2} - \nabla^2\right)\Phi^* - \varrho\omega^2\Phi^* + 2\mu\dfrac{\partial}{\partial z}\left(\dfrac{\partial^2}{\partial z^2} + \tau^2\right)\Psi^*. \\ \sigma^*_{rz} = 2\mu\dfrac{\partial^2\Phi^*}{\partial r\partial z} + \mu\dfrac{\partial}{\partial r}\left(2\dfrac{\partial^2}{\partial z^2} + \tau^2\right)\Psi^*. \end{cases}$$

The function Φ^* will be assumed in the form of a Hankel integral

$$(3.32)\qquad \Phi^* = \int_0^\infty (Ae^{-\lambda_1 z} + Be^{-\lambda_2 z})\, J_0(ar)\, da.$$

This integral is chosen so that the Eq. (3.26) is satisfied. The quantities λ_1, λ_2 are the roots of the equation

$$(3.33)\qquad \lambda^4 + [\sigma^2 - q(1+\varepsilon) - 2a^2]\,\lambda^2 + a^4 - a^2[\sigma^2 - q(+\varepsilon)] - q\sigma^2 = 0$$

and they are chosen so that the real part is positive. In this way we shall satisfy the condition $\overline{\overline{\Phi}}^* = 0$ at infinity.

The function Ψ^* will also be expressed by means of the Hankel integral

$$(3.34)\qquad \Psi^* = \int_0^\infty C(a)\, e^{-\nu z} J_0(ar)\, da.$$

where

$$\nu = \sqrt{a^2 - \tau^2}.$$

According to (3,28) we obtain:

$$(3.35)\qquad \Theta^* = \frac{1}{\vartheta_0}\int_0^\infty [A(\lambda_1^2 + \sigma^2 - a^2)\, e^{-\lambda_1 z} + B(\lambda_2^2 + \sigma^2 - a^2)\, e^{-\lambda_2 z}]\, J_0(ar)\, da.$$

The quatities A, B, C constituting functions of the parameter a will be determined using the boundary conditions (3.25). We obtain the following system of equations:

$$(3.36)\qquad \begin{cases} A(\lambda_1^2 + \sigma^2 - a^2) + B(\lambda_2^2 + \sigma^2 - a^2) = 0, \\ (A+B)(2\mu a^2 - \varrho\omega^2) - 2\mu\nu a^2 C = 0, \\ 2(A\lambda_1 + B\lambda_2) - (2\nu^2 - \tau^2)\, C - \dfrac{\vartheta_0 Q_0 a \Gamma_0}{\pi\varkappa(k_1^2 - k_2^2)} = 0. \end{cases}$$

gdzie

$$\Gamma_0 = e^{-\xi\sqrt{a^2+k_1^2}} - e^{-\xi\sqrt{a^2+k_2^2}},$$

Hence we find A, B, C:

$$A = \frac{2Q_0\vartheta_0\mu a^3 \Gamma_0}{\pi\varkappa(k_1^2 - k_2^2)} \frac{\nu(\lambda_2^2 + \sigma^2 - a^2)}{\Delta_1},$$

$$B = -\frac{\lambda_1^2 + \sigma^2 - a^2}{\lambda_2^2 + \sigma^2 - a^2} A, \qquad C = \frac{2\mu a^2 - \varrho\omega^2}{2\mu\nu a^2} \frac{\lambda_2^2 - \lambda_1^2}{(\lambda_2^2 + \sigma^2 - a^2)} A,$$

$$\Delta_1 = [\lambda_1(\lambda_2^2 + \sigma^2 - a^2) - \lambda_2(\lambda_1^2 + \sigma^2 - a^2)]\, 4\mu\nu a^2 - (2\mu a^2 - \varrho\omega^2)(\nu^2 + a^2)(\lambda_2^2 - \lambda_1^2).$$

Knowing Φ^*, Ψ^*, we can find the stresses σ_{ij}. The final stresses σ_{ij} will be obtained by adding $\bar{\sigma}_{ij}$ and $\bar{\bar{\sigma}}_{ij}$.

If the coupling of the temperature field and the stress field is not taken into consideration, we obtain for $\varepsilon = 0$

$$\lambda_1 = \sqrt{a^2 + q}, \quad \lambda^2 = \sqrt{a^2 - \sigma^2}, \quad \lambda_1^2 + \sigma^2 - a^2 = q + \sigma^2, \quad \lambda_2^2 + \sigma^2 - a^2 = 0.$$

Then, from the first of the Eqs. (3.36) it follows that $A = 0$, and from (3.35) $\Theta^* = 0$. Since for $\varepsilon = 0$ we have $k_1 = \sqrt{q}$, $k_2 = i\sigma$, therefore for the determination of B and C we obtain the following system of equations:

$$(2\mu a^2 - \varrho\omega^2) B - 2\mu\nu a^2 C = 0,$$

$$2B\sqrt{a^2 - \sigma^2} - (2a^2 - \tau^2) C - \frac{\vartheta_0 Q_0 a \Gamma_0}{\pi\varkappa(\sigma^2 + q)} = 0$$

whence

$$C = \frac{2\mu a^2 - \varrho\omega^2}{2\mu\nu a^2} B,$$

$$B = \frac{2\mu\vartheta_0 Q_0 a^3 \Gamma_0}{\pi\varkappa(\sigma^2 + q)} \frac{1}{4\mu\nu a^2\sqrt{a^2 - \sigma^2} - (2\mu a^2 - \varrho\omega)(2a^2 - \tau^2)}.$$

Let us consider the case of action of a point source of heat at the point $(0, 0, \xi)$ of the elastic semi-space assuming that in the $z = 0$ plane the following boundary conditions are valid:

$$\sigma_{zz} = 0, \quad \sigma_{xz} = 0, \quad \frac{\partial\Theta}{\partial z} = 0. \tag{3.37}$$

Let us apply the reflection method and let a positive heat source act at the point $(0, 0, \xi)$ and $(0, 0, -\xi)$ of the infinite space. They will provoke a temperature field $\bar{\Theta}$ and a state of stress $\bar{\sigma}_{ij}$. The boundary conditions $\bar{\sigma}_{rz} = 0$, $\partial\bar{\Theta}/\partial z = 0$ are satisfied. The stress $\bar{\sigma}_{zz}$ remains different from zero. The functions $\bar{\Phi}^*$ and $\bar{\Theta}^*$ are given by the equations:

$$\bar{\Phi}^* = \frac{Q_0\vartheta_0}{4\pi\varkappa(k_1^2 - k_2^2)} \left(\frac{e^{-k_1R_1} - e^{-k_2R_1}}{R_1} + \frac{e^{-k_1R_2} - e^{-k_2R_2}}{R_2} \right), \tag{3.38}$$

$$(3.39)\qquad \Theta^* = \frac{Q}{4\pi\varkappa(k_1^2-k_2^2)}\left[\frac{(\sigma^2+k_1^2)\,e^{-k_1R_1}-(\sigma^2+k_2^2)^{-k_2R_1}}{R_1}+\right.$$
$$\left.+\frac{(\sigma^2+k_1^2)\,e^{-k_1R_2}-(\sigma^2+k_2^2)\,e^{-k_2R_2}}{R_2}\right],$$

where

$$R_{1,2}=(r^2+(z\mp\zeta)^2)^{1/2},\qquad r=(x^2+y^2)^{1/2}.$$

The stress $\bar{\sigma}_{zz}^*$ is expressed by the equation:

$$(3.40)\qquad \bar{\sigma}_{zz}^* = -2\mu\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\bar{\Phi}^*-\varrho\omega^2\bar{\Phi}^* =$$
$$=-\frac{\vartheta_0 Q_0}{\pi^2\varkappa}\int_0^\infty\int_0^\infty\frac{(2\mu\alpha^2-\varrho\omega^2)}{F(\alpha,\gamma)}\,\alpha J_0(\alpha r)\cos\gamma z\cos\gamma\zeta\,d\alpha\,d\gamma.$$

For $z=0$ we obtain:

$$(3.41)\qquad \bar{\sigma}_{zz}^*(r,0)=$$
$$=\frac{\Theta_0\vartheta_0}{2\pi\varkappa(k_1^2-k_2^2)}\int_0^\infty(2\mu\alpha^2-\varrho\omega^2)\left[\frac{e^{-\zeta\sqrt{\alpha^2+k_1^2}}}{\sqrt{\alpha^2+k_1^2}}-\frac{e^{-\zeta\sqrt{\alpha^2+k_2^2}}}{\sqrt{\alpha^2+k_1^2}}\right]\alpha J_0(\alpha r)\,d\alpha.$$

To the state of stress $\bar{\sigma}_{ij}$, we should add the state $\bar{\bar{\sigma}}_{ij}$, so chosen that in the $z=0$ plane the following boundary conditions are satisfied:

$$(3.42)\qquad \bar{\bar{\sigma}}_{zz}^*+\bar{\sigma}_{zz}^*=0,\qquad \sigma_{rz}^*=0,\qquad \frac{d\Theta^*}{dz}=0.$$

The stresses $\bar{\bar{\sigma}}_{zz}^*$ and $\bar{\bar{\sigma}}_{rz}^*$ are given by the Eqs. (3.31) and the functions Φ^*, Θ^*, Ψ^* by the Eqs. (3.32), (3.35) and (3.34). The constants A, B, C appearing in these functions will be found from the boundary conditions (3.42). We obtain the following system for the determination of A, B, C:

$$(3.43)\qquad \begin{cases} (2\mu\alpha^2-\varrho\omega^2)(A+B)-2\mu\nu\alpha^2C+ \\ \qquad +\dfrac{\vartheta_0Q_0(2\mu\alpha^2-\varrho\omega^2)}{2\pi\varkappa(k_1^2-k_2^2)}\left(\dfrac{e^{-\zeta\sqrt{\alpha^2+k_1^2}}}{\sqrt{\alpha^2+k_1^2}}-\dfrac{e^{-\zeta\sqrt{\alpha^2+k_2^2}}}{\sqrt{\alpha^2+k_2^2}}\right)=0, \\ 2(A\lambda_1+B\lambda_2)-(2\alpha^2-\tau^2)C=0, \\ \lambda_1(\lambda_1^2+\sigma^2-\alpha^2)A+\lambda_2(\lambda_2^2+\sigma^2-\alpha^2)B=0. \end{cases}$$

At the point $(0,\zeta)$ of the elastic semi-space, let a linear source of heat act, normal to the xz-plane. Let us assume that the $z=0$ plane bounding the elastic semi-space is free from stress and, in addition, let the temperature in this plane be different from zero. We require that:

$$(3.44)\qquad \sigma_{zz}=0,\qquad \sigma_{xz}=0,\qquad \Theta=0\qquad \text{for}\qquad z=0.$$

Using the reflection method, we let a positive linear heat source act at the point $(0, \zeta)$ and a negative at the point $(0, -\zeta)$ of the infinite space.

These sources result in a temperature $\bar{\Theta}$ and stress $\bar{\sigma}_{ij}$. It can easily be found that in the $z=0$ plane we have $\bar{\Theta}=0$, $\sigma_{zz}=0$, but $\sigma_{rz} \neq 0$. The functions $\bar{\Phi}^*, \bar{\Theta}^*$ are given by the equations:

$$(3.45) \qquad \Phi^* = \frac{Q_0 \vartheta_0}{2\pi\varkappa (k_1^2 - k_2^2)} \left[K_0(k_1 r_1) - K_0(k_2 r_1) - K_0(k_1 r_2) + K_0(k_2 r_2)\right],$$

$$(3.46) \qquad \Theta^* = \frac{Q_0}{2\pi\varkappa (k_1^2 - k_2^2)} \left[(\sigma^2 + k_1^2)\left(K_0(k_1 r_1) - K_0(k_1 r_2)\right) - \right.$$
$$\left. - (\sigma^2 + k_2^2)\left(K_0(k_2 r_1) - K_0(k_2 r_2)\right)\right],$$

where $r_{1,2} = (x^2 + (z \mp \zeta)^2)^{1/2}$.

The function $\bar{\Phi}^*$ may be expressed also by means of the double Fourier integral

$$(3.47) \qquad \Phi^* = -\frac{\vartheta_0 Q_0}{\pi^2 \varkappa} \int_0^\infty \int_0^\infty \frac{\cos\beta x \cos\gamma z \left[\cos\gamma(z-\zeta) - \cos\gamma(z+\zeta)\right]}{F(\beta,\gamma)} \, d\beta d\gamma.$$

where

$$F(\beta,\gamma) = (\beta^2 + \gamma^2 + k_1^2)(\beta^2 + \gamma^2 + k_2^2).$$

The stress $[\bar{\sigma}_{xz}^*]_{z=0}$ will be found from the equation:

$$(3.48) \qquad [\bar{\sigma}_{xz}^*]_{z=0} = 2\mu \frac{\partial^2 \bar{\Phi}^*}{\partial x \partial z} =$$
$$= \frac{2\mu\vartheta_0 Q_0}{\pi^2 \varkappa} \int_0^\infty \frac{\beta}{(k_1^2 - k_2^2)} \left(e^{-\zeta\sqrt{\beta^2+k_1^2}} - e^{-\zeta\sqrt{\beta^2+k_2^2}}\right) \sin\beta x \, d\beta.$$

In order to suppress the stress $\bar{\sigma}_{ij}^*(x, 0)$, we shall add to the state of stress $\bar{\sigma}_{ij}^*$ the state of stress $\bar{\bar{\sigma}}_{ij}^*$ so chosen that in the $z=0$ plane the following boundary conditions are satisfied:

$$(3.49) \qquad \bar{\sigma}_{xz}^* + \bar{\bar{\sigma}}_{xz}^* = 0, \quad \bar{\bar{\sigma}}_{zz}^* = 0, \quad \bar{\bar{\Theta}}^* = 0.$$

The state $\bar{\bar{\sigma}}_{ij}$ will result in a temperature field $\bar{\bar{\Theta}}^*$, an elastic longitudinal wave and a transversal wave. The following two-dimensional wave equations should be satisfied:

$$(3.50) \qquad (\nabla_1^2 - q)(\nabla_1^2 + \sigma^2)\bar{\bar{\Phi}}^* - q\varepsilon\nabla^2\bar{\bar{\Phi}}^* = 0.$$

$$(3.51) \qquad (\nabla_1^2 + \tau^2)\,\mathrm{rot}\,\bar{\bar{\Psi}}^* = 0, \qquad \nabla_1^2 = \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial z^2}.$$

The temperature $\bar{\bar{\Theta}}^*$ is connected with the function $\bar{\bar{\Phi}}^*$ by the relation:

$$(3.52) \qquad (\nabla_1^2 + \sigma^2)\bar{\bar{\Phi}}^* = \vartheta_0 \bar{\bar{\Theta}}^*.$$

The stresses $\bar{\bar{\sigma}}^*_{xz}$ and $\bar{\bar{\sigma}}^*_{yz}$ are given by the equations:

$$(3.53)\qquad \begin{cases} \bar{\bar{\sigma}}^*_{xz} = 2\mu \dfrac{\partial^2 \bar{\bar{\Phi}}^*}{\partial x \partial z} + \mu\left(\dfrac{\partial^2}{\partial x^2} - \dfrac{\partial^2}{\partial z^2}\right) \Psi_1^*, \\ \bar{\bar{\sigma}}^*_{zz} = -2\mu \dfrac{\partial^2 \bar{\bar{\Phi}}^*}{\partial x^2} - \varrho\omega^2 \bar{\bar{\Phi}}^* + 2\mu \dfrac{\partial^2 \Psi_1^*}{\partial x \partial z}. \end{cases}$$

$$\operatorname{rot} \vec{\Psi}^* = [0, \Psi_1^*, 0], \quad u_1'' = -\frac{\partial}{\partial z} \Psi_1^*, \quad u_2'' = 0, \quad u_3'' = \frac{\partial}{\partial x} \Psi_1^*.$$

The functions $\bar{\bar{\Phi}}^*, \Psi_1^*, \bar{\bar{\Theta}}^*$ will be taken in the following form:

$$(3.54)\quad \bar{\bar{\Phi}}^*(x, z) = \int_0^\infty (Ae^{-\lambda_1 z} + Be^{-\lambda_2 z}) \cos \beta x \, d\beta,$$

$$(3.55)\quad \Psi_1^*(x, z) = \int_0^\infty Ce^{-\nu z} \sin \beta x \, d\beta,$$

$$(3.56)\quad \bar{\bar{\Theta}}^*(x, z) = \frac{1}{\vartheta_0} \int_0^\infty [A(\lambda_1^2 + \sigma^2 - \beta^2) e^{-\lambda_1 z} + B(\lambda_2^2 + \sigma^2 - \beta^2) e^{-\lambda_2 z}] \cos \beta x \, d\beta,$$

λ_1, λ_2 are the roots of the equation

$$\lambda^4 + [\sigma^2 - q(1+\varepsilon) - 2\beta^2] \lambda^2 + \beta^4 - \beta^2 [\sigma^2 - q(1+\varepsilon)] - q\sigma^2 = 0,$$

and are chosen in such a way that their real parts are positive. Next we have $\nu = \sqrt{\beta^2 - \tau^2}$.

The quantities $A(\beta), B(\beta), C(\beta)$ constituting functions of the parameter β will be found from the boundary conditions (3.49). Bearing in mind the relations (3.53) and (3.56), we obtain

$$(3.57)\quad \begin{cases} 2\mu\beta(\lambda_1 A + \lambda_2 B) - \mu(2\beta^2 - \tau^2) C + \dfrac{2\mu\vartheta_0 Q_0 \beta}{\pi\varkappa(k_1^2 - k_2^2)} \left(e^{-\xi\sqrt{\beta^2 + k_1^2}} - e^{-\xi\sqrt{\beta^2 + k_2^2}}\right) = 0, \\ (2\mu\beta^2 - \varrho\omega^2)(A + B) - 2\mu\beta\nu C = 0, \\ A(\lambda_1^2 + \sigma^2 - \beta^2) + B(\lambda_2^2 + \sigma^2 - \beta^2) = 0. \end{cases}$$

Knowing the quantities A, B, C and, therefore, the functions $\bar{\bar{\Phi}}^*, \Psi_1^*$, we can determine the state of stress $\bar{\bar{\sigma}}_{ij}$. Adding $\bar{\sigma}^*_{ij}$ to $\bar{\bar{\sigma}}^*_{ij}$, we obtain the stress σ^*_{ij}.

We can easily solve also the case where $\partial\Theta/\partial z = 0$ in $z = 0$ plane. We place linear positive heat sources at the points $(0, \zeta)$, $(0, -\zeta)$ of the space, satisfying the conditions $\partial\bar{\Theta}/\partial z = 0$, $\bar{\sigma}_{xz} = 0$. The stress $\bar{\sigma}_{zz}$ will be different from zero. To the state of stress $\bar{\sigma}_{ij}$ we add $\bar{\bar{\sigma}}_{ij}$ so chosen that in the $z = 0$ plane the following conditions are satisfied:

$$(3.58)\qquad \bar{\sigma}_{zz} + \bar{\bar{\sigma}}_{zz} = 0, \qquad \bar{\bar{\sigma}}_{xz} = 0, \qquad \frac{\partial\bar{\bar{\Theta}}}{\partial z} = 0.$$

For the functions $\bar{\bar{\Phi}}^*, \bar{\bar{\Theta}}^*, \Psi_1^*$ we shall take the integral expression (3.54) and (3.56), and the values A, B, C will be found from the conditions (3.58).

4. The State of Stress in an Infinite Space with a Cavity

Let the temperature on the surface $R=a$ bounding the cavity in the infinite elastic space be $\Theta(a,t)=\Theta_0 e^{i\omega t}$. The functions Φ^* and Θ^*, constituting the solution of the homogeneous equation (2.6) and the equation (2.2) may be expressed as:

$$\Phi^* = \frac{1}{R}(Ae^{-k_1 R} + Be^{-k_2 R}), \tag{4.1}$$

$$\Theta^* = \frac{1}{R\vartheta_0}\left[A(k_1^2+\sigma^2)e^{-k_1 R} + B(k_2^2+\sigma^2)e^{-k_2 R}\right]. \tag{4.2}$$

Let us assume that on the surface $R=a$, the stress σ_{RR} is equal to zero. To determine the constants A B we have two boundary conditions

$$\Theta^*(a)=\Theta_0, \qquad \sigma_{RR}^*(a) = -\left[\frac{4\mu}{R}\frac{d\Phi^*}{dR} + \varrho\omega^2\Phi^*\right]_{R=a} = 0, \tag{4.3}$$

leading to the system of equations:

$$\begin{cases} A(k_1^2+\sigma^2)e^{-k_1 a} + B(k_2^2+\sigma^2)e^{-k_2 a} = \Theta_0\vartheta_0 a, \\ Ae^{-k_1 a}\left[4\mu(1+ak_1) - a^2\omega^2\varrho\right] + Be^{-k_2 a}\left[4\mu(1+ak_2) - a^2\omega^2\varrho\right] = 0. \end{cases} \tag{4.4}$$

Solving this system of equations, we obtain:

$$A = \Theta_0 a\vartheta_0 \frac{m_2 e^{k_1 a}}{\Delta}, \qquad B = -\Theta_0 a\vartheta_0 \frac{m_1 e^{k_2 a}}{\Delta}, \tag{4.5}$$

where

$$\Delta = (k_1^2+\sigma^2)m_2 - (k_2^2+\sigma^2)m_1,$$

$$m_1 = 4\mu(1+ak_1) - a^2\omega^2\varrho, \qquad m_2 = 4\mu(1+ak_2) - a^2\omega^2\varrho.$$

Therefore:

$$\Phi^* = \frac{\Theta_0 a\vartheta_0}{R\Delta}\left[m_2 e^{-k_1(R-a)} - m_1 e^{-k_2(R-a)}\right], \tag{4.6}$$

$$\Theta^* = \frac{\Theta_0 a}{R\Delta}\left[(k_1^2+\sigma^2)m_2 e^{-k_1(R-a)} - (k_2^2+\sigma^2)m_1 e^{-k_2(R-a)}\right]. \tag{4.7}$$

Knowledge of the function Φ^* enables us to determine the state of stress. We obtain:

$$\begin{cases} \sigma_{RR} = -\frac{4\mu}{R}\frac{\partial\Phi}{\partial R} + \varrho\frac{\partial^2\Phi}{\partial t^2}, \\ \sigma_{\varphi\varphi} = \sigma_{\vartheta\vartheta} = -2\mu\left(\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r}\right) + \varrho\frac{\partial^2\Phi}{\partial t^2}. \end{cases} \tag{4.8}$$

In the particular case of $\varepsilon = 0$, therefore, if we do not take into account the coupling of the temperature field and the strain field, we have $k_1 = \sqrt{q}$, $k_2 = i\sigma$, $k_2^2 + \sigma^2 = 0$.

$$\Theta^* = \frac{\Theta_0 a}{R} e^{-(R-a)\sqrt{q}}, \tag{4.9}$$

$$\Phi^* = \frac{\Theta_0 a \vartheta_0}{(\sigma^2 + q) R} \left[\overline{m}_2 e^{-(R-a)\sqrt{q}} - \overline{m}_1 e^{-(R-a) i\sigma}\right], \tag{4.10}$$

where

$$\overline{m}_1 = 4\mu (1 + a\sqrt{q}) - a^2 \omega^2 \varrho, \qquad \overline{m}_2 = 4\mu (1 + ai\sigma) - a^2 \omega^2 \varrho.$$

In an analogous manner we can study the following boundary conditions:

a) $$\Theta^*(a) = \Theta_0, \qquad u_R^*(a) = \left[\frac{d\Phi^*}{dR}\right]_{R=a} = 0,$$

b) $$-\lambda \frac{dR}{dR} = W_0, \qquad \sigma_{RR}^*(a) = 0,$$

c) $$-\lambda \frac{dR}{dR} = W_0, \qquad u_R^*(a) = 0.$$

Let us study now the other type of problem. Let on the boundary surface $R = a$ of the cavity act the pressure $p(a, t) = p_0 e^{i\omega t}$. We shall represent here also the function Φ^* by means of the Eq. (4.1)$_1$ and function Θ^* by the Eq. (4.2). Let us assume moreover that $\Theta^* = 0$ for $R = a$. The boundary conditions take the following form:

$$\Theta^*(a) = 0, \qquad \sigma_{RR}^*(a) = -\left[\frac{4\mu}{R}\frac{d\Phi^*}{dR} + \varrho\omega^2 \Phi^*\right]_{R=a} = -p_0. \tag{4.11}$$

The constants A, B are determined by the equations:

$$A = \frac{p_0 a^3 e^{k_1 a}}{\Delta}, \qquad B = -\frac{p_0 a^3 e^{k_2 a}}{\Delta} \frac{k_1^2 + \sigma^2}{k_2^2 + \sigma^2}. \tag{4.12}$$

Thus:

$$\Phi^* = -\frac{p_0 a^3}{\Delta R}\left[e^{-(R-a)k_1} - \frac{k_1^2 + \sigma^2}{k_2^2 + \sigma^2} e^{-(R-a)k_2}\right](k_2^2 + \sigma^2), \tag{4.13}$$

$$\Theta^* = -\frac{p_0 a^3}{\vartheta_0 \Delta R}\left[e^{-(R-a)k_1} - e^{-(R-a)k_2}\right](k_2^2 + \sigma^2). \tag{4.14}$$

If the coupling of the temperature field and the strain field is not taken into account $(\varepsilon = 0)$, we have:

$$\Phi^* = \frac{p_0 a^3}{\overline{m}_2 R} e^{-(R-a)i\sigma}, \qquad \Theta^* = 0. \tag{4.15}$$

In an analogous manner, we can solve three further problems with the boundary conditions:

a) $$\Theta^*(a) = 0\,, \qquad u_R^*(a) = u_0 = \text{const}\,,$$

b) $$\frac{d\Theta^*(a)}{dR} = 0\,, \quad u_{RR}^*(a) = -p_0 = \text{const}\,,$$

c) $$\frac{d\Theta^*(a)}{dR} = 0\,, \qquad u_R^*(a) = u_0 = \text{const}\,.$$

5. The State of Stress and the Temperature Field Due to the Action of Dilatation Nuclei

In a region Γ of the infinite space let the initial dilatation $e(P, t)$ be given. In this case, the system of displacement equations, after the introduction of the displacement function (1.3), will become the following system of three equations:

$$\left(\nabla^2 - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}\right)\Phi = \vartheta_0\Theta - \xi e\,, \tag{5.1}$$

$$\left(\nabla^2 - \frac{1}{\varkappa}\frac{\partial}{\partial t}\right)\Theta - \eta\frac{\partial}{\partial t}\nabla^2\Phi = 0\,, \tag{5.2}$$

$$\left(\nabla^2 - \frac{1}{c_2^2}\frac{\partial^2}{\partial t^2}\right)\operatorname{rot}\Psi = 0\,. \tag{5.3}$$

In the case of an infinite space and the action of the dilatation $e(P, t)$, we have $\Psi = 0$. If $e(P,t) = e^{i\omega t}e^*(P)$, we have $\Phi(P,t) = e^{i\omega t}\Phi^*(P)$, $\Theta(P,t) = e^{i\omega t}\Theta^*(P)$.

The Eqs. (5.1) and (5.2) become:

$$(\nabla^2 + \sigma^2)\Phi^* = \vartheta_0\Theta^* - \xi e^*\,, \tag{5.4}$$

$$(\nabla^2 - q)\Theta^* - q\eta'\nabla^2\Phi^* = 0\,, \qquad \xi = \frac{1}{3}\,\frac{3\lambda + 2\mu}{\lambda + 2\mu}\,. \tag{5.5}$$

By eliminating from (5.4) (5.5) first the function Θ^* and then the function Φ^* we obtain the respective equations:

$$(\nabla^2 + \sigma^2)(\nabla^2 - q)\Phi^* - q\varepsilon\nabla^2\Phi^* = -\xi(\nabla^2 - q)e^*, \tag{5.6}$$

$$(\nabla^2 + \sigma^2)(\nabla^2 - q)\Theta^* - q\varepsilon\nabla^2\Theta^* = -q\eta'\xi\nabla^2 e^*. \tag{5.7}$$

In the infinite space let a concentrated dilatation nucleus $e^*(R) = e_0\delta(R)$ act, where $\delta(R)$ is the D i r a c function. In a way analogous to that used in the Sec. 2, we find that:

$$\Phi^* = \frac{e_0\xi}{4\pi R(k_1^2 - k_2^2)}\left[(k_1^2 - q)e^{-k_1 R} - (k_2^2 - q)e^{-k_2 R}\right], \tag{5.8}$$

$$\Theta^* = \frac{e_0\xi\eta'}{4\pi R(k_1^2 - k_2^2)}\left(k_1^2 e^{-k_1 R} - k_2^2 e^{-k_2 R}\right). \tag{5.9}$$

The knowledge of the function Φ^* enables us to determine the stresses from the Eqs. (2.12).

In the particular case where $\eta' = 0$, that is if the coupling between the temperature and the strain field is disregarded, we obtain the known result [11]

$$\Phi = \frac{e_0 \xi}{4\pi R} e^{i(\omega t - R\sigma)}, \qquad \Theta^* = 0. \tag{5.10}$$

For dilatation nuclei evenly spaced along the z-axis $[e^*(r) = e_0\,\delta(r)]$, we have,

$$\Phi^* = \frac{e_0 \xi}{2\pi (k_1^2 - k_2^2)} \left[(k_1^2 - q)\,\mathrm{K}_0(k_1 r) - (k_2^2 - q)\,\mathrm{K}_0(k_2 r)\right], \tag{5.11}$$

$$\Theta^* = \frac{e_0 \xi \eta'}{2\pi (k_1^2 - k_2^2)} \left[k_1^2\,\mathrm{K}_0(k_1 r) - k_2^2\,\mathrm{K}_0(k_2 r)\right]. \tag{5.12}$$

If $\varepsilon = 0$, we have:

$$\Phi = \frac{e_0 \xi}{2\pi} \mathrm{K}_0(i\sigma r)\, e^{i\omega t}, \qquad \Theta = 0. \tag{5.13}$$

Finally, in the case of dilatation nuclei evenly spaced in the $x = 0$ plane $[e^*(x) = e_0\,\delta(x)]$, we obtain:

$$\Phi^* = \frac{e_0 \xi}{2(k_1^2 - k_2^2)} \left(\frac{k_1^2 - q}{k_1} e^{-k_1 x} - \frac{k_2^2 - q}{k_2} e^{-k_2 x} \right), \tag{5.14}$$

$$\Theta^* = \frac{e_0 \xi \eta'}{2(k_1^2 - k_2^2)} \left(k_1 e^{-k_1 x} - k_2 e^{-k_2 x}\right). \tag{5.15}$$

In the case when the coupling between the temperature field and the strain field is disregarded, we obtain the familiar result [4]:

$$\Phi = \frac{e_0 \xi}{2} e^{i(\omega t - x\sigma)}, \qquad \Theta = 0. \tag{5.16}$$

The state of stress due to the action of dilatation nuclei in an elastic semi-space may be found in a manner analogous to what was done in § 3.

6. The Action of a Concentrated Force in an Infinite Space

Let a concentrated force $P(t) = P_0\, e^{i\omega t}$ act at the origin of cylindrical coordinates in the direction of the z-axis. We are concerned with longitudinal and transversal waves and the temperature field Θ in the elastic space.

The following wave equation is to be solved:

$$(\nabla^2 + \sigma^2)(\nabla^2 - q)\,\Phi^* - q\varepsilon \nabla^2 \Phi^* = 0, \tag{6.1}$$

$$(\nabla^2 + \tau^2)\operatorname{rot}\Psi = \Theta, \qquad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \tag{6.2}$$

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[4] See for instance [11].

Knowing the function Φ^*, we can find the temperature field from the equation:

(6.3)
$$\Theta^* = \frac{1}{\vartheta_0} (\nabla^2 + \sigma^2) \Phi^*.$$

The functions Φ^*, Ψ^* will be taken in the form of the following Hankel's integrals:

(6.4)
$$\begin{cases} \Phi^* = \int\limits_0^\infty (Ae^{-\lambda_1 z} + Be^{-\lambda_2 z}) J_0(ar)\, da, & z > 0, \\ \Phi^* = \int\limits_0^\infty (A' e^{\lambda_1 z} + B' e^{\lambda_2 z}) J_0(ar)\, da, & z < 0; \end{cases}$$

(6.5)
$$\begin{cases} \Psi^* = \int\limits_0^\infty Ce^{-\nu z} J_0(ar)\, da, & z > 0, \\ \Psi^* = \int\limits_0^\infty C' e^{\nu z} J_0(ar)\, da, & z < 0, \end{cases}$$

From the Eq. (6.3), we obtain:

(6.6)
$$\begin{cases} \Theta^* = \frac{1}{\vartheta_0} \int\limits_0^\infty (A n_1 e^{\lambda_1 z} + B n_2 e^{\lambda_2 z}) J_0(ar)\, da, & z > 0, \\ \Theta^* = \frac{1}{\vartheta_0} \int\limits_0^\infty (A' n_1 e^{\lambda_1 z} + B' n_2 e^{\lambda_2 z}) J_0(ar)\, da, & z < 0, \end{cases}$$

$$n_{1,2} = \lambda_{1,2}^2 + \sigma^2 - a^2.$$

The quantities λ_1, λ_2 are the roots of the Eq. (3.33) assuming that they are the roots whose real parts are positive. Next $\nu = \sqrt{a^2 - \tau^2}$.

The constants $A, A', \ldots, C'$ will be found from the following boundary conditions in the $z = 0$ plane:

(6.7)
$$\begin{cases} [u_r^*]_{+0} - [u_r^*]_{-0} = 0, & [w^*]_{+0} - [w^*]_{-0} = 0, \\ [\Theta^*]_{+0} - [\Theta^*]_{-0} = 0, & \left[\frac{\partial \Theta^*}{\partial z}\right]_{+0} - \left[\frac{\partial \Theta^*}{\partial z}\right]_{-0} = 0, \\ [\sigma_{rz}^*]_{+0} - [\sigma_{rz}^*]_{-0} = 0, & [\sigma_{zz}^*]_{+0} - [\sigma_{zz}^*]_{-0} + \frac{P_0}{2\pi r} \delta(r) = 0, \\ \frac{\delta(r)}{2\pi r} = \frac{1}{2\pi} \int\limits_0^\infty a J_0(ar)\, da. \end{cases}$$

The first two conditions warrant the continuity of displacements in the $z = 0$ plane, the two next — the continuity of temperature and its gradient in the $z = 0$ plane. The penultimate condition postulates the continuity of the shear stress σ_{rz}^*, and the last expresses the discontinuity of the stress σ_{zz}^* due to the introduction of the concentrated force.

Using the equations for displacements e_r^*, w^*, and the Eqs. (3.31), and bearing in mind the Eqs. (6.4) and (6.6) we obtain from the boundary conditions (6.7) the following system of equations:

$$(6.8)\quad \begin{cases} (A-A')+(B-B')-\nu(C+C')=0\,, \\ \lambda_1(A+A')\lambda_2+(B+B')-\alpha^2(C-C')=0\,, \\ (A-A')n_1+(B-B')n_2=0\,, \\ 2\lambda_1(A+A')+2\lambda_2(B+B')-(2\alpha^2-\tau^2)(C-C')=0\,, \\ (2\mu\alpha^2-\varrho\omega^2)(A-A'+B-B')-2\mu\alpha^2\nu(C+C')+\dfrac{P_0\alpha}{2\pi}=0\,. \end{cases}$$

Solving this, we obtain:

$$(6.9)\quad \begin{cases} A=-A'=\dfrac{P_0\alpha}{4\pi}\dfrac{n_2}{(n_2-n_1)\varrho\omega^2}\,,\quad B=-B'=-\dfrac{P_0\alpha}{4\pi}\dfrac{n_1}{(n_2-n_1)\varrho\omega^2}\,, \\ C=C'=\dfrac{P_0\alpha}{4\pi\nu\varrho\omega^2}\,. \end{cases}$$

Thus the functions Φ^*, Θ^*, Ψ^* are determined in the entire elastic space. Knowledge of the function Φ^*, Ψ^* enables the determination of the stress components (for instance σ_{rz}^*, σ_{zz}^* from (3.31). If the influence of coupling between the strain field and the temperature field ($\varepsilon=0$) is disregarded, then: $\lambda_1^2=\alpha^2+q$, $\lambda_2^2=\alpha^2+\sigma^2$, $n_1=q+\sigma^2$, $n_2=0$. In this case, we obtain $A=A'=0$.

$$(6.10)\qquad B=-B'=\frac{P_0\alpha}{4\pi\mu\tau^2}\,,\qquad C=C'=\frac{P_0\alpha}{4\pi\mu\nu\tau^2}\,,$$

according to the known results [5].

Since $A=A'=0$ and $n_2=0$, it is seen from (6.6) that $\Theta^*=0$, in the entire elastic space.

Let a load of intensity $P(t)=e^{i\omega t}P_0$, uniformly distributed along the y-axis of a rectangular system of coordinates, act in the positive sense of the z-axis. The stress and temperature field will be functions of the variables x and z only. The equations (6.1) and (6.2) should be satisfied, being replaced by $\nabla_1^2=(\partial^2/\partial x^2)+(\partial^2/\partial z^2)$.

The functions Φ^*, Ψ^* and Θ^* will be expressed by means of the following Fourier integrals:

$$(6.11)\qquad \begin{cases} \Phi^*=\displaystyle\int_0^\infty (Ae^{-\lambda_1 z}+Be^{-\lambda_2 z})\cos\beta x\,d\beta\,, & z>0\,, \\ \Phi^*=\displaystyle\int_0^\infty (A'e^{\lambda_1 z}+B'e^{\lambda_2 z})\cos\beta x\,d\beta\,, & z<0\,; \end{cases}$$

[5] See for instance [11], p. 42.

$$(6.12)\qquad \begin{cases} \Psi^* = \int\limits_0^\infty C e^{-\nu z} \sin \beta x d\beta, & z > 0, \\ \Psi^* = \int\limits_0^\infty C' e^{\nu z} \sin \beta x d\beta, & z < 0; \end{cases}$$

$$(6.13)\qquad \begin{cases} \Theta^* = \dfrac{1}{\vartheta_0} \int\limits_0^\infty (A n_1 e^{-\lambda_1 z} + B n_2 e^{-\lambda_2 z}) \cos \beta x d\beta, & z > 0, \\ \Theta^* = \dfrac{1}{\vartheta_0} \int\limits_0^\infty (A' n_1 e^{\lambda_1 z} + B' n_2 e^{\lambda_2 z}) \cos \beta x d\beta, & z < 0, \\ n_{1,2} = \lambda_{1,2}^2 + \sigma^2 - \beta^2, \end{cases}$$

where λ_1, λ_2 are the roots of the equation

$$\lambda^4 + [\sigma^2 - q(1+\varepsilon) - 2\beta^2]\lambda^2 + \beta^4 - \beta^2[\sigma^2 - q(1+\varepsilon)] - q\sigma^2 = 0,$$

assuming that the real parts of these roots are positive. The values $A, A', \dots, C'$ will be obtained from the following boundary conditions expressing the continuity of displacements, temperature, and stresses in the cross-section $z = 0$.

$$(6.14)\qquad \begin{cases} [u^*]_{+0} - [u^*]_{-0} = 0, & [w^*]_{+0} - [w^*]_{-0} = 0, \\ [\Theta^*]_{+0} - [\Theta^*]_{-0} = 0, & \left[\dfrac{\partial\Theta^*}{\partial z}\right]_{+0} - \left[\dfrac{\partial\Theta^*}{\partial z}\right]_{-0} = 0, \\ [\sigma_{xz}^*]_{+0} - [\sigma_{xz}^*]_{-0} = 0, & [\sigma_{zz}^*]_{+0} - [\sigma_{zz}^*]_{-0} + P\delta(x) = 0, \\ \delta(x) = \dfrac{1}{\pi} \int\limits_0^\infty \cos \beta x d\beta. \end{cases}$$

From the boundary conditions (6.14), we obtain the system of equations:

$$(6.15)\qquad \begin{cases} (A - A' + B - B')\beta - \nu(C + C') = 0, \\ \lambda_1(A + A') + \lambda_2(B + B') + \beta(C - C') = 0, \\ (A - A')n_1 + (B - B')n_2 = 0, \\ \lambda_1 n_1 (A + A')\lambda_2 n_2 (B + B') = 0, \\ 2\mu\beta[\lambda_1(A + A') + \lambda_2(B + B')] - \mu(\beta^2 + \nu^2)(C - C') = 0, \\ (2\mu\beta^2 - \varrho\omega^2)(A - A' + B - B') - 2\mu\nu\beta(C + C') + \dfrac{P_0}{\pi} = 0. \end{cases}$$

Solving this equation, we obtain:

$$(6.16)\qquad \begin{cases} A = -A' = \dfrac{P_0}{2\pi} \dfrac{n_2}{(n_2 - n_1)\varrho\omega^2}, \quad B = -B' = -\dfrac{P_0}{2\pi} \dfrac{n_1}{(n_2 - n_1)\varrho\omega^2}, \\ C = C' = \dfrac{P_0 \beta}{2\pi\nu\varrho\omega^2}. \end{cases}$$

In the particular case $\varepsilon=0$, we have $n_2=0$, therefore $\Theta^*=0$, $A=-A'=0$ and

$$B=-B'=\frac{P_0}{2\pi\mu\tau^2}, \quad C=C'=\frac{P_0\beta}{2\pi\nu\mu\tau^2}.$$

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Streszczenie

PEWNE ZAGADNIENIA DYNAMICZNE TERMOSPRĘŻYSTOŚCI

W pracy rozpatrzono dwa typy zagadnień. Pierwsze zagadnienie, to wyznaczenie naprężeń wywołanych działaniem skupionego, liniowego i płaskiego źródła ciepła w przestrzeni i półprzestrzeni sprężystej przy założeniu sprzężenia pola odkształceń i temperatury. W przestrzeni sprężystej mamy do czynienia jedynie ze zmodyfikowanymi falami cieplnymi i sprężystymi podłużnymi. W przypadku półprzestrzeni sprężystej, w której działa liniowe lub skupione źródło ciepła obok fal podłużnych występują poprzeczne fale sprężyste.

Przy sposobności wyznaczenia stanu naprężenia dla półprzestrzeni podano rozwiązanie zmodyfikowanego zagadnienia Lamba, postulując sprzężenie pól odkształcenia i temperatury. W ustępie czwartym podano

rozwiązanie zagadnienia rozprzestrzeniania się naprężeń w przestrzeni z pustką kulistą ogrzaną na powierzchni $R = a$.

Drugi typ zagadnień to wyznaczenie stanu naprężenia wywołanego w przestrzeni i półprzestrzeni sprężystej działaniem okresowo zmieniającego się w czasie jądra dylatacji. Wyznaczono również stan naprężenia wywołany działaniem sił okresowo zmieniających się w czasie przyłożonych na powierzchni $R = a$ ograniczającej pustkę. Wreszcie podano rozwiązanie dla przypadku siły o intensywności $P(t) = e^{i\omega t} P_0$ działającej w nieograniczonej przestrzeni sprężystej.

Резюме

НЕКОТОРЫЕ ДИНАМИЧЕСКИЕ ЗАДАЧИ ТЕРМОУПРУГОСТИ

Рассматриваются два типа задач. Первая это — определение напряжений, вызванных действием сосредоточенного, линейного и плоского источника тепла в упругом пространстве и полупространстве при предположении сопряжении поля деформаций и температурного поля. В упругом простарнстве встречаемся лишь с модифицированными термическими и с упругими продольными волнами. В случае упругого полупространства, в котором действует линейный или сосредоточенный источник тепла наряду с продольными волнами выступают поперечные упругие волны.

По способности определения напряженного состояния для полупространства дается решение модифицированной задачи Ламба, постулируется сопряжение полей деформаций и температурных полей. В пкт. 4 дается решение вопроса распространения напряжений в пространстве со сферической пустотой нагретой на поверхности $R = a$.

Второй тип задач это определение напряженного состояния, вызванного в упругом пространстве и полупространстве, вызванного действием сил изменяющихся периодически во времени, приложенных к поверхности $R = a$, ограничивающей пустоту. В заключение дается решение для случая силы интенсивностью $P(t) = e^{i\omega t} P_0$, действующей в неограниченном упругом пространстве.

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