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CERTAIN STABILITY PROBLEMS OF RECTANGULAR PLATES

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1. The object of this investigation is to determine critical loads of rectangular plates simply supported on two opposite edges and having different boundary conditions along the two remaining edges.

It is assumed that the forces $q_x = h\,\sigma_x$, $q_y = h\sigma_y$ and $q_{xy} = h\tau_{xy}$ (h denotes the plate thickness) act in the plate and are, in general, functions of the variables x and y, being first determined from the solution of the problem of plane stress. Our problem consists in reaching an accurate solution of the stability problem of rectangular plates of this type. The method adopted is as follows. The differential equation of the problem is replaced by the equivalent integral equation replaced in turn by an infinite system of secular linear equations. The buckling condition is the principal determinant of this system set equal to zero. Then the critical load of the plate is directly obtained from the lowest eigenvalue of the kernel of the integral equation.

First, the following fundamental problem will be solved. Consider a rectangular plate simply supported on the edges $x = 0$, $x = a$ and clamped at the remaining edges. Such a system is equivalent to a plate strip with a periodic load (with the period b) q_x, q_y and q_{xy}, and additional forces $r(x)$ and $\bar{r}(x)$ perpendicular to the plate and acting along the segments $y = \pm\, ib/2$ ($i = 0,1,2, \ldots$). The value of these additional forces is so selected that the deflection w of the plate vanishes along the line of their action, as also (for reasons of symmetry) the slope $\partial w/\partial y$ of the deflection surface. The differential equation of this problem has the well-known form

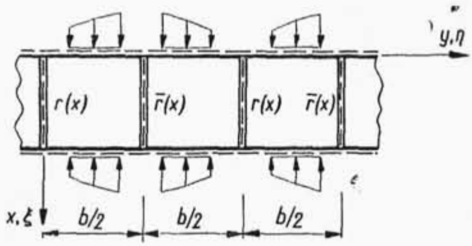

Fig. 1

$$(1.1)\quad N\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) = \\ = -\left(q_x\frac{\partial^2 w}{\partial x^2} + q_y\frac{\partial^2 w}{\partial y^2} + 2\,q_{xy}\frac{\partial^2 w}{\partial x\,\partial y}\right) + R(x,y) + \bar{R}(x,y),$$

where N denotes the flexural rigidity of the plate and $R(x, y)$ or $\bar{R}(x, y)$ the external load comprising periodically distributed linear loads $r(x)$ or $\bar{r}(x)$.

The solution of differential equation (1.1) can be expressed in the form

$$(1.2)\quad w(x, y) = -\iint_{\Omega} w(x, y; \xi, \eta)\left(q_\xi \frac{\partial^2 w}{\partial \xi^2} + q_\eta \frac{\partial^2 w}{\partial \eta^2} + 2\, q_{\xi\eta} \frac{\partial^2 w}{\partial \xi \partial \eta}\right) d\xi\, d\eta + \\ + w_R(x, y) + w_{\bar{R}}(x, y);$$

here $w(x, y; \xi, \eta)$ denotes the Green function of the Eq. (1.1). This function represents physically the displacement w of a point of the plate with the coordinates (x, y), due to the action of concentrated forces (Fig. 2); $w_R(x, y)$ and $w_{\bar{R}}(x, y)$ denote the deflection surfaces due to the action of the additional forces $r(x)$ and $\bar{r}(x)$ (Fig. 3), respectively. These quantities will be determined under the assumption of $q_x = q_y = q_{xy} = 0$.

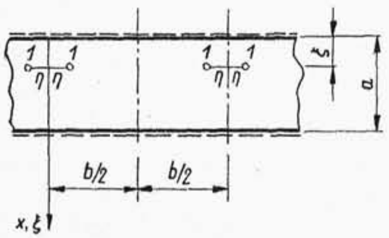

Fig. 2

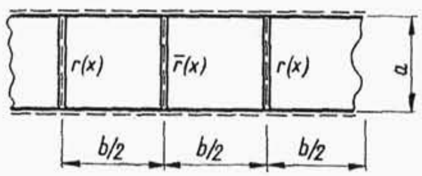

Fig. 3

The load by the concentrated forces, according to Fig. 2, can be represented in the form of the infinite series

$$(1.3)\quad p(x, y) = \frac{4}{ab} \sum_n \sin \alpha_n \xi \sin \alpha_n x + \\ + \frac{8}{ab} \sum_m \sum_n \sin \alpha_n \xi \sin \alpha_n x \cos \beta_m \eta \cos \beta_m y,$$

where

$$\alpha_n = \frac{n\pi}{a}, \qquad \beta_m = \frac{2m\pi}{b}.$$

Similarly, the load by the additional forces $r(x)$ and $\bar{r}(x)$, according to Fig. 3, can be represented as

$$(1.4)\quad R(x, y) + \bar{R}(x, y) = \frac{1}{b} \sum_n r_n \sin \alpha_n x + \frac{2}{b} \sum_m \sum_n r_n \sin \alpha_n x \cos \beta_m y + \\ + \frac{1}{b} \sum_n \bar{r}_n \sin \alpha_n x + \frac{2}{b} \sum_m \sum_n \bar{r}_n (-1)^m \sin \alpha_n x \cos \beta_m y,$$

where r_n and $\bar{r}_n$ are the Fourier expansion coefficients for the functions $r(x)$ and $\bar{r}(x)$ in the x-direction. The Green function $\bar{w}(x, y; \xi, \eta)$ should satisfy the differential equation

(1.5) $$\Delta\Delta\, w(x, y; \xi, \eta) = p(x, y).$$

Assuming that

$$\bar{w}(x, y, \xi, \eta) = \sum_n a_n \sin \alpha_n x + \sum_m \sum_n b_{nm} \sin \alpha_n x \cos \beta_m y,$$

we obtain directly

(1.6) $$\bar{w}(x, y, \xi, \eta) = \frac{4}{ab} \sum_n \frac{\sin \alpha_n \xi \sin \alpha_n x}{\Delta_n} + \frac{8}{ab} \sum_n \sum_m \frac{\sin \alpha_n \xi \sin \alpha_n x \cos \beta_m \eta \cos \beta_m y}{\Delta_{nm}},$$

where

$$\Delta_n = N \alpha_n^4, \qquad \Delta_{nm} = N(\alpha_n^2 + \beta_m^2)^2.$$

The functions $w_R(x, y)$ and $w_{\bar{R}}(x, y)$ are obtained from the equations

(1.7) $$\begin{cases} w_R(x,y) = \int_0^a \int_0^b R(\xi, \eta)\, \bar{w}(x, y, \xi, \eta)\, d\xi\, d\eta = \\ \qquad = \frac{1}{b} \sum_n \frac{r_n \sin \alpha_n x}{\Delta_n} + \frac{2}{b} \sum_n \sum_m \frac{r_n \sin \alpha_n x \cos \beta_m y}{\Delta_{nm}}, \\ w_{\bar{R}}(x,y) = \int_0^a \int_0^b \bar{R}(\xi, \eta)\, \bar{w}(x, y, \xi, \eta)\, d\xi\, d\eta = \end{cases}$$

$$= \frac{1}{b} \sum_n \frac{\bar{r}_n \sin \alpha_n x}{\Delta_n} + \frac{2}{b} \sum_n \sum_m \frac{r_n (-1)^m \sin \alpha_n x \cos \beta_m y}{\Delta_{nm}}.$$

We now apply the Green transformation to the surface integral appearing in the Eq. (1.2):

(1.8) $$-\iint_\Omega w(x, y, \xi, \eta) \left(q_\xi \frac{\partial^2 w}{\partial \xi^2} + q_\eta \frac{\partial^2 w}{\partial \eta^2} + 2\, q_{\xi\eta} \frac{\partial^2 w}{\partial \xi\, \partial \eta} \right) d\xi\, d\eta =$$

$$= \int_s \left\{ \left[w(\xi, \eta) \frac{\partial \bar{Q}_n}{\partial n} - \bar{Q}_n \frac{\partial w(\xi, \eta)}{\partial n} \right] - \left[w(\xi, \eta) \frac{\partial \bar{Q}_{ns}}{\partial s} - \bar{Q}_{ns} \frac{\partial w(\xi, \eta)}{\partial s} \right] \right\} ds -$$

$$- \iint_\Omega w(\xi, \eta) \left[\frac{\partial^2 (\bar{w}\, q_\xi)}{\partial \xi^2} + \frac{\partial^2 (\bar{w}\, q_\eta)}{\partial \eta^2} + 2 \frac{\partial^2 (\bar{w}\, q_{\xi\eta})}{\partial \xi\, \partial \eta} \right] d\xi\, d\eta,$$

the notations being as follows:

$$\bar{Q}_n = \bar{w}(x, y, \xi, \eta)\, (q_\xi \cos^2 \varphi + q_\eta \sin^2 \varphi + 2\, q_{\xi\eta} \sin \varphi \cos \varphi),$$

$$\bar{Q}_{ns} = \bar{w}(x, y, \xi, \eta)\, [(q_\xi - q_\eta) \sin \varphi \cos \varphi + q_{\xi\eta} (\cos^2 \varphi - \sin^2 \varphi)].$$

The surface integral is taken over the rectangle with the edges $x = 0$, a; $y = 0$, $b/2$.

The curvilinear integral in the Eq. (1.8) is, in our case, exactly equal to zero, since $w(\xi, \eta) = 0$ and $\overline{w}(x, y, \xi, \eta) = 0$ at the edges of the plate considered. Thus, we pass to the surface integral appearing in the right-hand member of that equation. Substituting the expression (1.6) for $\overline{w}$ we obtain, after the necessary calculations and rearrangement,

$$\frac{\partial^2 (\overline{w}\, q_\xi)}{\partial \xi^2} + \frac{\partial^2 (\overline{w}\, q_\eta)}{\partial \eta^2} + 2\frac{\partial^2 (\overline{w}\, q_{\xi\eta})}{\partial \xi\, \partial \eta} = \frac{4}{ab} \sum_n \frac{\sin \alpha_n x}{\Delta_n} \times$$

$$\times \left[\sin \alpha_n \xi \left(\frac{\partial^2 q_\xi}{\partial \xi^2} + \frac{\partial^2 q_\eta}{\partial \eta^2} + 2\frac{\partial^2 q_{\xi\eta}}{\partial \xi\, \partial \eta} \right) + 2\,\alpha_n \cos \alpha_n \xi \left(\frac{\partial q_\xi}{\partial \xi} + \frac{\partial q_{\xi\eta}}{\partial \eta} \right) - \right.$$

$$\left. - q_\xi\, \alpha_n^2 \sin \alpha_n \xi \right] + \frac{8}{ab} \sum_n \sum_m \frac{\sin \alpha_n x \cos \beta_m y}{\Delta_{nm}} \left[\sin \alpha_n \xi \cos \beta_m \eta \times \right.$$

$$\times \left(\frac{\partial^2 q_\xi}{\partial \xi^2} + \frac{\partial^2 q_\eta}{\partial \eta^2} + 2\frac{\partial^2 q_{\xi\eta}}{\partial \xi\, \partial \eta} \right) + 2\,\alpha_n \cos \alpha_n \xi \cos \beta_m \eta \left(\frac{\partial q_\xi}{\partial \xi} + \frac{\partial q_{\xi\eta}}{\partial \eta} \right) -$$

$$- 2\,\beta_m \sin \alpha_n \xi \sin \beta_m \eta \left(\frac{\partial q_\eta}{\partial \eta} + \frac{\partial q_{\xi\eta}}{q\, \xi} \right) - q_\xi\, \alpha_n^2 \sin \alpha_n \xi \cos \beta_m \eta -$$

$$\left. - q_\eta\, \beta_m^2 \sin \alpha_n \xi \cos \beta_m \eta - 2\, q_{\xi\eta}\, \alpha_n \beta_m \cos \alpha_n \xi \sin \beta_m \eta \right].$$

We can now use the familiar equations of equilibrium for a plane state of stress:

$$\frac{\partial \sigma_\xi}{\partial \xi} + \frac{\partial \tau_{\xi\eta}}{\partial \eta} + \varrho\, X = 0, \qquad \frac{\partial \tau_{\xi\eta}}{\partial \xi} + \frac{\partial \sigma_\eta}{\partial \eta} + \varrho\, Y = 0.$$

Assuming that the mass distribution is of the potential type, that is,

$$\varrho\, X = -\frac{\partial \Phi}{\partial \xi}, \qquad \varrho\, Y = -\frac{\partial \Phi}{\partial \eta},$$

we obtain the conditions

$$(1.9) \qquad \left\{ \begin{aligned} & \frac{\partial q_\xi}{\partial \xi} + \frac{\partial q_{\xi\eta}}{\partial \eta} = h \left(\frac{\partial \sigma_\xi}{\partial \xi} + \frac{\partial \tau_{\xi\eta}}{\partial \eta} \right) = h \frac{\partial \Phi}{\partial \xi}, \\ & \frac{\partial q_{\xi\eta}}{\partial \xi} + \frac{\partial q_\eta}{\partial \eta} = h \left(\frac{\partial \tau_{\xi\eta}}{\partial \xi} + \frac{\partial \sigma_\eta}{\partial \eta} \right) = h \frac{\partial \Phi}{\partial \eta}, \\ & \frac{\partial^2 q_\xi}{\partial \xi^2} + 2\frac{\partial^2 q_{\xi\eta}}{\partial \xi\, \partial \eta} + \frac{\partial^2 q_\eta}{\partial \eta^2} = h\, \Delta \Phi. \end{aligned} \right.$$

Using these equations we can write the integral equation (1.2) in the form

$$(1.10)\quad w(x,y)=\int_0^a\int_0^{b/2} w(\xi,\eta)\Bigg[\frac{4}{ab}\sum_n \frac{\sin\alpha_n x}{\Delta_n}\Big(h\,\Delta\Phi\sin\alpha_n\xi+2h\frac{\partial\Phi}{\partial\xi}\alpha_n\cos\alpha_n\xi-$$

$$-q_\xi\alpha_n^2\sin\alpha_n\xi\Big)+\frac{8}{ab}\sum_n\sum_m\frac{\sin\alpha_n x\cos\beta_m y}{\Delta_{nm}}\Big(h\,\Delta\Phi\sin\alpha_n\xi\cos\beta_m\eta+$$

$$+2h\frac{\partial\Phi}{\partial\xi}\alpha_n\cos\alpha_n\xi\cos\beta_m\eta-2h\frac{\partial\Phi}{\partial\eta}\beta_m\sin\alpha_n\xi\sin\beta_m\eta-q_\xi\alpha_n^2\sin\alpha_n\xi\cos\beta_m\eta-$$

$$-q_\eta\beta^2\sin\alpha_n\xi\cos\beta_m\eta-2q_{\xi\eta}\alpha_n\beta_m\cos\alpha_n\xi\sin\beta_m\eta\Big)\Bigg]d\xi\,d\eta+$$

$$+w_R(x,y)+w_{\bar R}(x,y).$$

If $\partial\Phi/\partial\xi\equiv\partial\Phi/\partial\eta\equiv\Delta\Phi\equiv 0$, the body forces are also equal to zero. This being the case, we can rewrite the integral equation considered in the more compact form:

$$(1.11)\quad w(x,y)=\int_0^a\int_0^{b/2} w(\xi,\eta)\Bigg\{\frac{4}{ab}q_\xi\sum_n\frac{\alpha_n^2\sin\alpha_n x\sin\alpha_n\xi}{\Delta_n}+$$

$$+\frac{8}{ab}\sum_m\sum_n\frac{\sin\alpha_n x\cos\beta_m y}{\Delta_{nm}}\big[(q_\xi\alpha_n^2+q_\eta\beta_m^2)\sin\alpha_n\xi\cos\beta_m\eta+$$

$$+2q_{\xi\eta}\alpha_n\beta_m\cos\alpha_n\xi\sin\beta_m\eta\big]\Bigg\}d\xi\,d\eta+w_R(x,y)+w_{\bar R}(x,y).$$

We have thus obtained a Fredholm non-homogeneous integral equation of the second kind.

For the solution of this equation, we assume that the deflection of the plate has the form

$$(1.12)\qquad w(x,y)=\sum_n A_n\sin\alpha_n x+\sum_n\sum_m B_{nm}\sin\alpha_n x\cos\beta_m y.$$

Substituting the expressions (1.12) and (1.7) in the Eq. (1.11), we obtain, after certain simple transformations, the following relations between the coefficients A_n, B_{nm}, r_n and $\bar r_n$:

$$(1.13)\quad \left\{\begin{aligned} A_n&=\frac{4}{ab\,\Delta_n}\Big(\sum_j A_j a_{jn}+\sum_j\sum_k B_{jk}b_{jkn}\Big)+\frac{r_n}{b\,\Delta_n}+\frac{\bar r_n}{b\,\Delta_n},\\ B_{nm}&=\frac{8}{ab\,\Delta_{nm}}\Big(\sum_j A_j c_{jnm}+\sum_j\sum_k B_{jk}d_{jknm}\Big)+\frac{2r_n}{b\,\Delta_{nm}}+\frac{2\bar r_n(-1)^m}{b\,\Delta_{nm}}.\end{aligned}\right.$$

The notations introduced are as follows:

$$
(1.14)\quad \begin{cases}
a_{jn} = \int\limits_0^a \int\limits_0^{b/2} q_\xi(\xi,\eta)\, \alpha_n^2 \sin \alpha_j \xi \sin \alpha_n \xi \, d\xi \, d\eta, \\
b_{jkn} = \int\limits_0^a \int\limits_0^{b/2} q_\xi(\xi,\eta)\, \alpha_n^2 \sin \alpha_j \xi \sin \alpha_n \xi \cos \beta_k \eta \, d\xi \, d\eta, \\
c_{jnm} = \int\limits_0^a \int\limits_0^{b/2} K_{nm}(\xi,\eta) \sin \alpha_j \xi \, d\xi \, d\eta, \\
d_{jknm} = \int\limits_0^a \int\limits_0^{b/2} K_{nm}(\xi,\eta) \sin \alpha_j \xi \cos \beta_k \eta \, d\xi \, d\eta, \\
K_{nm}(\xi,\eta) = [q_\xi(\xi,\eta)\, \alpha_n^2 + q_\eta(\xi,\eta)\, \beta_m^2] \sin \alpha_n \xi \cos \beta_m \eta + \\
\qquad + 2\, q_{\xi\eta}(\xi,\eta)\, \alpha_n \beta_m \cos \alpha_n \xi \sin \beta_m \eta.
\end{cases}
$$

In order to determine the coefficients A_n, B_{nm}, r_n and $\bar{r}_n$, two further equations can be used. Bearing in mind the boundary conditions $w(x, 0) = w(x, b/2) = 0$, we obtain these equations, from the Eq. (1.12), in the form

$$
(1.15)\qquad A_n + \sum_m B_{nm} = 0, \qquad A_n + \sum_m (-1)^m B_{nm} = 0.
$$

The Eqs. (1.13) and (1.15) represent a complete system which suffices to determine the deformation of the plate and the critical loads. This system can, in certain case, be reduced to a simpler form containing the B_{nm} coefficients only.

We pass now to the investigation of several particular cases. If we consider the case of a plate loaded symmetrically in relation to the straight line parallel to the x-axis and passing through the middle of the plate, the assumption of $\bar{r}_n = 0$ and the rejection of the second condition (1.15) will suffice. Thus, we obtain the system of equations,

$$
(1.16)\quad \begin{cases}
A_n = \dfrac{4}{ab\, \Delta_n} \left(\sum\limits_j A_j a_{jn} + \sum\limits_j \sum\limits_k B_{jk} b_{jkn} \right) + \dfrac{r_n}{b\, \Delta_n}, \\
B_{nm} = \dfrac{8}{ab\, \Delta_{nm}} \left(\sum\limits_j A_j c_{jnm} + \sum\limits_j \sum\limits_k B_{jk} d_{jknm} \right) + \dfrac{2\, r_n}{b\, \Delta_{nm}}, \\
A_n + \sum\limits_m B_{nm} = 0,
\end{cases}
$$

for a plate with the dimensions $a \cdot b$ (see Fig. 1). The system of equations (1.16) can, by successive elimination of the coefficients A_n and r_n, be reduced to the more compact form

$$
(1.17)\qquad \Delta_{nm} B_{nm} + 2\, \Delta_n \sum_m B_{nm} = \frac{8}{ab} \sum_j \sum_k B_{jk} (a_{jn} - b_{jkn} - c_{jnm} + d_{jknm}).
$$

Starting from the Eqs. (1.13) and (1.15), we can also investigate the case of a simply supported plate strip acted on by periodically distributed forces q_x, q_y and q_{xy} (Fig. 1).

If the coefficients r_n and $\bar{r}_n$ in the Eqs. (1.13) are disregarded, and the two equations (1.15) excluded, we obtain directly

$$(1.18)\qquad \begin{cases} A_n = \dfrac{4}{ab\,\Delta_n}\left(\sum\limits_j A_j\, a_{jn} + \sum\limits_j \sum\limits_k B_{jk}\, b_{jkn}\right), \\ B_{nm} = \dfrac{8}{ab\,\Delta_{nm}}\left(\sum\limits_j A_j\, c_{jnm} + \sum\limits_j \sum\limits_k B_{jk} d_{jknm}\right). \end{cases}$$

In this case, attention should be drawn to the fact that the corresponding curvilinear integral in the Eq. (1.8) is equal to zero, since $w\,(x, b/2) \neq 0$.

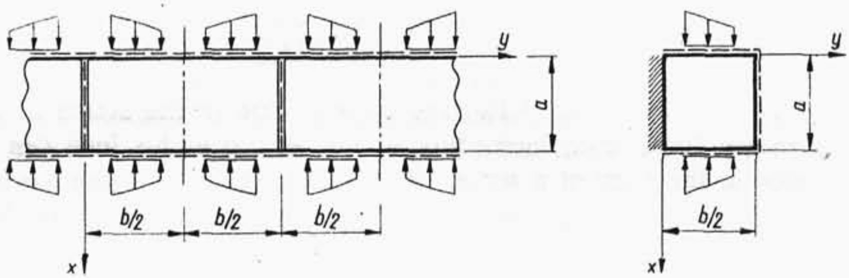

Fig. 4

This can easily be demonstrated by using the condition that $q_{xy} = 0$ along these segments $(x = 0,\ x = b)$ in consequence of the symmetry of load.

We now pass to another case. Consider an $a \cdot b/2$ plate simply supported on three edges, the edge $y = 0$ being clamped (Fig. 4). This case can be obtained from an infinite plate strip subjected to the action of periodic forces q, and to the supplementary forces $r\,(x) \cdot (-1)^i$ along the lines $y = ib$ $(i = 0,1, \ldots)$.

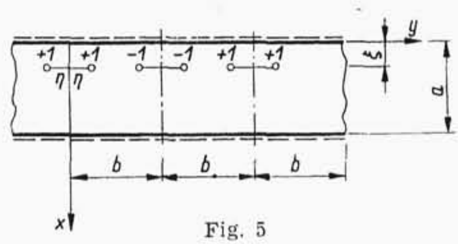

Fig. 5

The integral equation has in this case the form

$$(1.19)\qquad w\,(x,y) = -\iint\limits_{\Omega} \bar{\bar{w}}(x,y,\xi,\eta)\left(q_\xi \frac{\partial^2 w}{\partial \xi^2} + q_\eta \frac{\partial^2 w}{\partial \eta^2} + 2\, q_{\xi\eta} \frac{\partial^2 w}{\partial \xi\, \partial \eta}\right) d\xi\, d\eta + w_r\,(x,y).$$

The Green function w is obtained by determining the deflection of the infinite plate strip loaded by the periodic forces $+1$, -1 according to Fig. 5. This problem consists in the solution of the equation

$$N \Delta \Delta\, \bar{\bar{w}}\,(x, y, \xi, \eta) = p\,(x, y, \xi, \eta),$$

where

$$p(x, y, \xi, \eta) = \frac{8}{ab} \sum_m \sum_n \sin a_n \xi \sin a_n x \cos \bar{\beta}_m \eta \cos \bar{\beta}_m y,$$

and

$$a_n = \frac{n\pi}{a}, \qquad \bar{\beta}_m = \frac{(2m-1)\pi}{b} \qquad (m, n = 1, 2, \ldots).$$

Assuming $\bar{\bar{w}}$ in the form,

$$\bar{\bar{w}} = \sum_n \sum_m \bar{\bar{w}}_{nm} \sin a_n x \cos \bar{\beta}_m y,$$

we can easily find the Green function for the Eq. (1.20):

$$\bar{\bar{w}}(x, y, \xi, \eta) = \frac{8}{ab} \sum_m \sum_n \frac{\sin a_n \xi \sin a_n x \cos \bar{\beta}_m \eta \cos \bar{\beta}_m y}{\Delta_{nm}}.$$

Similarly we obtain the deflection $w_r(x, y)$ due to the action of the supplementary forces $r(x)$, according to Fig. 4. Since this load can be represented in the form of a series

$$\frac{2}{b} \sum_m \sum_n r_n \sin a_n x \cos \beta_m y,$$

we obtain

$$w_r(x, y) = \frac{2}{b} \sum_m \sum_n \frac{r_n \sin a_n x \cos \bar{\beta}_m y}{\Delta_{nm}}. \tag{1.20}$$

If Green transformation is applied, as before, to the double integral, and the corresponding curvilinear integral disregarded, we obtain a Fredholm non-homogeneous integral equation of the second kind:

$$w(x, y) = -\int_0^a \int_0^{b/2} \bar{\bar{w}}(x, y, \xi, \eta) \left(q_\xi \frac{\partial^2 w}{\partial \xi^2} + q_\eta \frac{\partial^2 w}{\partial \eta^2} + 2 q_{\xi\eta} \frac{\partial^2 w}{\partial \xi \partial \eta} \right) d\xi \, d\eta + w_r(x, y). \tag{1.21}$$

After a transformation of the integrand, and using the conditions (1.9) and (1.10) we solve the equation (1.21) by substituting

$$w(x, y) = \sum_m \sum_n A_{nm} \sin a_n x \cos \beta_m y.$$

Thus we obtain the system of equations

$$A_{nm} = \frac{8}{ab\,\Delta_{nm}} \sum_j \sum_k A_{jk} d_{jknm} + \frac{2 r_n}{b\,\Delta_{nm}}. \tag{1.22}$$

The second system of equations will be obtained from the boundary condition $w(x, ib) = 0$. From the Eq. (1.20), we obtain:

$$\sum_m A_{nm} = 0. \tag{1.23}$$

The Eqs. (1.22) and (1.23) enable us completely to solve the problem under consideration.

We can pass to a certain particular case, assuming $r(x) = 0$ and rejecting the condition (1.23). Thus, we obtain the system of equations

$$A_{nm} = \frac{8}{ab\,\Delta_{nm}} \sum_j \sum_k A_{jk}\, d_{jknm}, \tag{1.24}$$

already obtained for a plate simply supported on all edges and having the dimensions a and b, [1].

The above considerations based on the reduction of the determination of the critical loads to that of the zeros of an infinite determinant, produce, except in certain particular cases, enormous difficulties. Practical calculations must therefore be confined to a few elements of an infinite matrix and of a determinant of the third or fourth order.

Considerable difficulties are also encountered in the general case of plate load if we want to take into consideration the accurate values of the stresses σ_x, σ_y and τ_{xy}. These stresses, calculated from a plane state of stress, are usually represented by infinite series or by Fourier integrals. In many simple cases of load, where the forces q are constant over certain regions of the plate, the calculation is, however, considerably simplified.

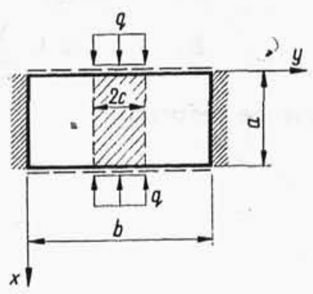

Fig. 6

2. In the second part of this paper, we shall present a series of simple examples some of which have already been solved in other ways by various authors. They serve as illustrations of the procedure described.

(a) A plate clamped along two opposite edges and acted on in the x-direction by a load uniformly distributed over the segment $2c$ (Fig. 6).

From the equation (1.4) we obtain, with the simplified assumption of $q_x = q_{xy} = 0$, $q_y = q$ for $b/2 - c < y < b/2 + c$ and $q_y = 0$ over the remaining part of the plate:

$$a_{jn} = \int\limits_{b/2-c}^{b/2} \int\limits_0^a q\, \alpha_n^2 \sin \alpha_j \xi \sin \alpha_n \xi \, d\xi\, d\eta = q\, \alpha_n^2 \frac{ac}{2}\, \delta_{jn},$$

$$b_{jkn} = \int\limits_{b/2-c}^{b/2} \int\limits_0^a q\, \alpha_n^2 \sin \alpha_j \xi \sin \alpha_n \xi \cos \beta_k \eta \, d\xi\, d\eta = \frac{q\, \alpha_n^2\, a}{2} \, \frac{\sin \beta_k c}{\beta_k}\, \delta_{jn},$$

$$c_{jnm} = q\,\frac{\alpha_n^2\, a}{2}\,\frac{\sin\beta_m c}{\beta_m}\,\delta_{jn},$$

$$d_{jknm} = q\,\frac{\alpha_n^2\, a}{2}\left[\frac{\sin(\beta_k+\beta_m)c}{\beta_k+\beta_m} - \frac{\sin(\beta_k-\beta_m)c}{\beta_k-\beta_m}\right]\delta_{jn},$$

where δ_{jn} is the Kronecker symbol.

The system of equations (1.17) therefore takes the form

$$\Delta_{nm} B_{nm} = \sum_k B_{nk}\left\{\frac{4\,q\,c\,\alpha_n^2}{b}\left[1 - 2\,\frac{\sin\beta_k c}{\beta_k} + \left(\frac{\sin(\beta_k+\beta_m)c}{(\beta_k+\beta_m)} - \frac{\sin(\beta_k-\beta_m)c}{(\beta_k-\beta_m)}\right)\right] - 2\,\Delta_n\right\}. \tag{2.1}$$

Owing to the symbol δ_{ik}, the series appearing in the Eq. (2.1) is very simple.

Further simplification is obtained in the case of $2c = b$, that is for a plate subjected to a uniform pressure.

Then,

$$a_{jn} = q\,\alpha_n^2\,\frac{ab}{4}\,\delta_{jn}, \qquad b_{jkn} = c_{jnm} = 0, \qquad d_{jknm} = q\,\frac{ab}{8}\,\alpha_n^2\,\delta_{jn}\,\delta_{km}.$$

The system of equations

$$B_{nm}\,\Delta_{nm} + 2\,\Delta_n \sum_k B_{nk} = \frac{8}{ab}\left(q\,\alpha_n^2\,\frac{ab}{4}\sum_k B_{nk} + q\,\alpha_n^2\,\frac{8}{ab}\,B_{nm}\right)$$

can be reduced to

$$\frac{B_{nm}}{2\,(q\,\alpha_n^2 - \Delta_n)} = \frac{\sum_k B_{nk}}{\Delta_{nm} - q\,\alpha_n^2},$$

and

$$\frac{1}{2\,(q\,\alpha_n^2 - \Delta_n)} = \sum_m \frac{1}{\Delta_{nm} - q\,\alpha_n^2}. \tag{2.2}$$

Hence the critical value of the load q can be obtained with a desired degree of accuracy. Thus, for instance, using only two terms of the sum (2.2), we obtain the critical load for $b/a = 2$ as equal to $q = 7.77\ N\pi^2/b^2$ — instead of $7.69\ N\pi^2/b^2$ as obtained by Timoshenko, [2].

(b) A plate subjected to concentrated forces (Fig. 7). We assume an approximate force distribution in the plate taking $q_{xy} = q_y = 0$, $q_x = P/2 \times [\delta(y-c) + \delta(y-b+c)]$, $\delta(y)$ as denoting the so-called Dirac function[1].

[1] The meaning of this symbol is explained by the equation

$$\int_a^b f(x)\,\delta(x-c)\,dx = f(c), \qquad a < c < b.$$

Under these assumptions we have

$$a_{jn} = \frac{Pa}{4} \alpha_n^2 \delta_{jn},$$

$$b_{jkn} = \frac{Pa}{4} \alpha_n^2 \cos \beta_k c\, \delta_{jn},$$

$$c_{jnm} = \frac{Pa}{4} \alpha_n^2 \cos \beta_m c\, \delta_{jn},$$

$$d_{jknm} = \frac{Pa}{4} \alpha_n^2 \cos \beta_k c \cos \beta_m c\, \delta_{jn}.$$

Our system of equations takes the following form:

$$\Delta_{nm} B_{nm} + 2 \Delta_n \sum_k B_{nk} = \frac{2 P \alpha_n^2}{b} \sum_k B_{nk} (1 - \cos \beta_k c)(1 - \cos \beta_m c).$$

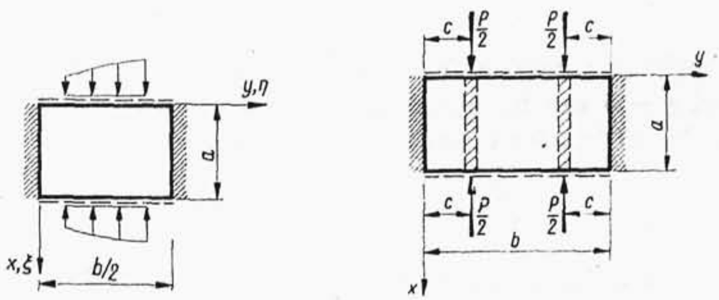

Fig. 7

This system can be transformed as follows:

$$(2.3) \qquad \bar{c} = \frac{2 P \alpha_n^2}{b} s_1 \bar{\bar{c}} - 2 \Delta_n s_0 \bar{c}, \qquad \bar{\bar{c}} = \frac{2 P \alpha_n^2}{b} s_2 \bar{c} - 2 \Delta_n s_1 \bar{c},$$

with the notations

$$(2.4) \qquad \begin{cases} \sum_m B_{nm} = \bar{c}, & \sum_m B_{nm} (1 - \cos \beta_m c) = c, \\ s_0 = \sum_m \frac{1}{\Delta_{nm}}, & s_1 = \sum_m \frac{1 - \cos \beta_m c}{\Delta_{nm}}, \qquad s_2 = \sum_m \frac{(1 - \cos \beta_m c)^2}{\Delta_{nm}} \end{cases}$$

From the system of equations (2.3) we can easily find the value of the critical force P in a closed form:

$$P = \frac{b}{2 \alpha_n^2} \frac{1 + 2 s_0 \Delta_n}{2 \Delta_n (s_0 s_2 - s_1^2) + s_2}.$$

In the particular case of $c = b/2$ (the load of the plate consisting in one force P), we obtain, after calculation of the sums (2.4),

$$P_{cr} = \frac{4 N \pi n}{a} \frac{\operatorname{sh} \varepsilon \operatorname{ch} \varepsilon + \varepsilon}{\operatorname{sh}^2 \varepsilon - \varepsilon^2}, \qquad \varepsilon = \frac{n \pi b}{2a}.$$

If we pass to the limit for $b \to \infty$, we obtain the known Sommerfeld solution, $P_{cr} = 4 N \pi / a$, [3]. In a similar manner, the solution in a closed form can be obtained for a plate loaded by an arbitrary number of $2l$ concentrated forces symmetrically arranged. This problem leads to the calculation of the determinant of the order $(l+1)$.

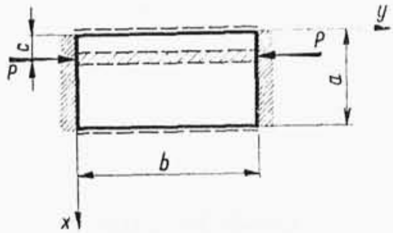

Fig. 8

(c) A plate loaded by concentrated forces according to Fig. 8.

Assuming that $q_x = q_{xy} = 0$ and $q_y = P\,\delta(y-c)$, we obtain the system of equations

$$B_{nm}\,\Delta_{nm} + 2\,\Delta_n \sum_m B_{nm} = = \frac{2\,P\,\beta_m^2}{a} \sum_j B_{jm} \sin \alpha_n c \sin \alpha_j c.$$

The buckling condition for a plate uniformly loaded at the edges $y = 0$ and $y = b$ can be found in a convenient way. In this case, we calculate the critical force from the equation

$$\frac{1}{2\,\Delta_n} + \sum_m \frac{1}{\Delta_{nm} - \mathrm{q}\,\beta_m^2} = 0,$$

obtained by a suitable transformation of the system (1.17).

(d) An infinite plate strip loaded in a periodic manner according to Fig. 9.

The values of the coefficients in the Eq. (1.14) for a simplified stress distribution, similar to the above, are

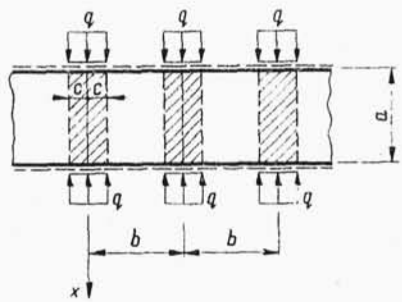

Fig. 9

$$a_{jn} = \mathrm{q}\,\alpha_n^2 \frac{ac}{2}\,\delta_{jn},$$

$$\mathrm{b}_{jkn} = \mathrm{q}\,\alpha_n^2 \frac{a}{2}\,\frac{\sin \beta_k c}{\beta_k}\,\delta_{jn},$$

$$c_{jnm} = \mathrm{q}\,\alpha_n^2 \frac{a}{2}\,\frac{\sin \beta_m c}{\beta_m}\,\delta_{jn},$$

$$d_{jknm} = \mathrm{q}\,\alpha_n^2 \frac{a}{4}\left[\frac{\sin(\alpha_k + \alpha_m)c}{\alpha_k + \alpha_m} - \frac{\sin(\alpha_k - \alpha_m)c}{\alpha_k - \alpha_m}\right]\delta_{jn}.$$

The equations (1.18) can be reduced to the unique equation

$$B_{nm} = \frac{8}{ab\,\Delta_{nm}} \sum_k B_{nk} \left(d_{jknm} + 4\,\frac{\mathrm{b}_{jkn}\,c_{jnm}}{ab\,\Delta_n - 4\,a_{jn}}\right). \tag{2.5}$$

This undergoes a further simplification in the limit case of $c \to 0$ where $\lim\limits_{c\to 0} 2qc = P$. We then obtain the known example of a plate strip loaded by concentrated forces. Thus, the expression for the critical force is obtained in the closed form:

$$P_{cr} = \frac{N b\, \alpha_n^2}{1 + 2 \sum\limits_m \dfrac{1}{\left[1 + m^2 \left(\dfrac{a}{b}\right)^2\right]^2}} = 4 \frac{N \pi n}{a} \frac{\mathrm{sh}^2 \varepsilon}{\varepsilon + \mathrm{sh}\, \varepsilon\, \mathrm{ch}\, \varepsilon}, \qquad \varepsilon = \frac{n \pi b}{2a}.$$

(e) A plate simply supported on three edges and uniformly loaded in one direction by a load q (Fig. 10).

The d_{jknm} component in the Eq. (1.21) is

$$d_{jknm} = q\, \alpha_n^2 \int\limits_0^a \sin \alpha_n \xi \sin \alpha_j \xi \, d\xi \int\limits_0^{b/2} \cos \beta_m \eta \cos \beta_k \eta \, d\eta,$$

$$d_{jknm} = q\, \alpha_n^2 \frac{ab}{8} \delta_{jn} \delta_{km}.$$

In consequence, the system of equations (1.21) is reduced to

$$(2.6) \qquad A_{nm} = \frac{q\, \alpha_n^2}{\Delta_{nm}} A_{nm} + \frac{2}{b\, \Delta_{nm}} r_n.$$

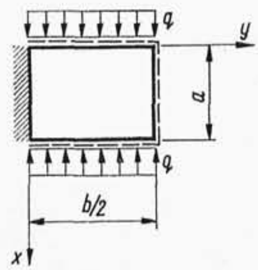

Fig. 10

The Eq. (2.6), together with the condition (1.23), enables us to represent the buckling condition in the form

$$(2.7) \qquad \sum_m \frac{1}{q\, \alpha_n^2 - \Delta_{nm}} = 0.$$

The calculation of the infinite sum (2.7) leads to the transcendental equation known from the technical literature on the subject under consideration (see, for instance, [2]):

$$u_1 \operatorname{tg} \frac{\pi u_2}{2} = u_2 \operatorname{tgh} \frac{\pi u_1}{2},$$

where

$$u_{1,2} = \sqrt{\lambda \pm c^2}, \qquad c = n \frac{b}{2}, \qquad \lambda = c \sqrt{\frac{q b^2}{N \pi^2}},$$

(f) Finally, it is desired to indicate the possibility of using the above method in a series of problems concerning plates loaded in a non-typical manner. As an example, consider a plate simply supported on all edges and loaded according to Figs. 11a and 11b. In both cases, the state of stress is determined as follows:

for Fig. 11*a*,

$$q_y = q_{xy} = 0, \qquad q_x = \begin{cases} 0 & \text{for} \quad x < a/2, \\ P\delta(y - b/2) & \text{for} \quad x > a/2; \end{cases}$$

for Fig. 11*b*,

$$q_y = q_{xy} = 0, \qquad q_x = \begin{cases} 0 & \text{for} \quad x < a/2, \\ q & \text{for} \quad x > a/2. \end{cases}$$

The Eqs. (1.24) will take, in the first case, the form

$$A_{nm} = \frac{4}{ab\,\Delta_{nm}} \sum_j \sum_k A_{jk}\, e_{jknm}, \tag{2.8}$$

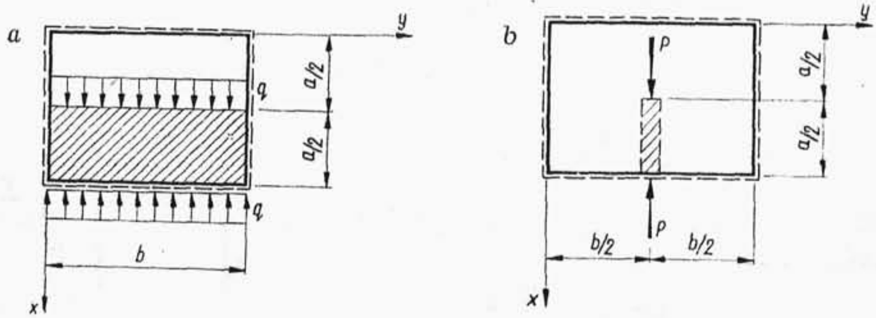

Fig. 11

where

$$e_{jknm} = \begin{cases} 0 \text{ for } j+n=2i \quad \text{and} \quad j \neq n, \\ \dfrac{P}{a_j + a_n} (-1)^{m+k+\frac{j+n+1}{2}} \quad \text{for} \quad j+n = 2i-1, \\ \dfrac{Pa}{4} (-1)^{m+k} \quad \text{for} \quad j = n. \end{cases}$$

In the second case (Fig. 11*b*), we obtain the system of equations

$$A_{nm} = \frac{4}{ab\,\Delta_{nm}} \sum_j A_{jm}\, e_{jnm}, \tag{2.9}$$

where

$$e_{jnm} = \begin{cases} 0 \text{ for } j+n = 2i,\ j \neq n, \\ \dfrac{qb}{2(a_j + a_n)} (-1)^{\frac{j+n+1}{2}} \text{ for } j+n = 2i-1, \\ \dfrac{ab}{8} \text{ for } j = n. \end{cases}$$

Confining ourselves to a few terms of the sums appearing in the relations (2.8) and (2.9), we can obtain, by setting the principal determinant of the corresponding systems of equations equal to zero, approximate values for the critical loads.

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[2] S. Timoshenko, *Theory of Elastic Stability*, New York and London 1956.

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Streszczenie

O PEWNYCH ZAGADNIENIACH STATECZNOŚCI PŁYT PROSTOKĄTNYCH

Przedmiotem pracy jest wyznaczenie krytycznych wartości obciążenia płyt prostokątnych, podpartych swobodnie na dwóch krawędziach i spełniających różne warunki na pozostałych brzegach. Składowe stanu naprężenia w płycie mogą być w ogólnym przypadku dowolnymi funkcjami współrzędnych x i y.

Warunek równowagi płyty podano w postaci równania całkowego Fredholma (1.11) wzgl. (1.21). Poszukiwanie wartości własnej jądra tego równania sprowadzić można, stosując podstawienia trygonometryczne typu (1.12), do znajdywania nietrywialnych rozwiązań układów równań algebraicznych (1.13), (1.15) albo (1.22), (1.23).

W szeregu przypadków szczególnych uzyskano ścisłe rozwiązanie problemu stateczności, a obciążenia krytyczne wyrażono w postaci zamkniętej. W przypadkach pozostałych ograniczyć się należy do uwzględnienia tylko części nieskończonego wyznacznika otrzymanych układów równań. Uzyskane w ten sposób rozwiązania przybliżone wykazują zadowalającą zgodność z rozwiązaniami ścisłymi, uzyskanymi na innej drodze.

Резюме

О НЕКОТОРЫХ ПРОБЛЕМАХ УСТОЙЧИВОСТИ ПРЯМОУГОЛЬНЫХ ПЛАСТИНОК

В работе рассматривается определение критических значений нагрузки прямоугольных пластинок, свободно опертых на двух краях и удовлетворяющих различным условиям на остальных. Компоненты напряженного состояния в пластинке могут быть в общем случае произвольными функциями координат x и y.

Условие равновесия пластинки приводится в виде интегрального уравнения Фредгольма (1.11) или (1.21). Поиски собственного значения ядра этого уравнения можно, применяя тригонометрические подстановки типа (1.12), свести к нахождению нетривиальных решений систем алгебраических уравнений (1.13) и (1.15) или (1.22) и (1.23).

В целом ряду случаев получены точные решения проблемы устойчивости, а критические нагрузки выражены в замкнутом виде. В остальных случаях следует ограничиться учетом только части бесконечного детерминанта системы уравнений. Полученные таким образом приближенные решения удовлетворительно согласуются с точными решениями, полученными другим путем.

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