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APPLICATION OF INTEGRAL TRANSFORMS IN THE THEORY OF ELASTICITY

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DISTORTION PROBLEMS OF ELASTICITY

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1. Fundamental relations and equations

Consider an isotropic and homogeneous elastic body, subjected to permanent strain ε_{ij}^{0} depending on the position $\underset{\sim}{x}$. This strain can arise in metals in exceeding the yield limit or during changes occurring in a heat working. A special case of distortion is the temperature strain $\varepsilon_{ij}^{0} = \alpha_t \delta_{ij}$. Here α_t denotes the coefficient of linear thermal expansion and $T = T_1 - T_0$ is the temperature increase. $T_1(\underset{\sim}{x})$ is the absolute temperature at point $\underset{\sim}{x}$ and T_0 = const is the absolute temperature of the natural state.

We assume that the strain ε_{ij}^{0} is of the same order as the elastic strain. The introduction of the initial strain ε_{ij}^{0} into the body produces a state of elastic strain $\varepsilon_{ij}^{!}$ and the state of the stress σ_{ij}.

The total strain ε_{ij} consists of two parts, the initial strain ε^0_{ij} and the elastic strain ε'_{ij}.

$$\varepsilon_{ij} = \varepsilon^0_{ij} + \varepsilon'_{ij} \tag{1.1}$$

The elastic strain ε'_{ij} is a linear function of the stress

$$\varepsilon'_{ij} = 2\mu'\sigma_{ij} + \lambda'\delta_{ij}\sigma_{kk}. \tag{1.2}$$

Substituting (1.2) into (1.1) and solving the latter equations for the stresses, we obtain

$$\sigma_{ij} = 2\mu(\varepsilon_{ij}-\varepsilon^0_{ij}) + \lambda\delta_{ij}(\varepsilon_{kk}-\varepsilon^0_{kk}), \tag{1.3}$$

where μ,λ are the Lamé constants, and

$$2\mu' = \frac{1}{2\mu}, \quad \lambda' = -\frac{\lambda}{2\mu(3\lambda+2\mu)}.$$

The total strain ε_{ij} can be expressed in terms of the displacement vector $\underset{\sim}{u}$ as follows:-

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j}+u_{j,i}). \tag{1.4}$$

If the stress σ_{ij} given by (1.3) is introduced into the equilibrium equations

$$\sigma_{ji,j} = 0, \tag{1.5}$$

and the relations (1.4) are taken into account, then the system of three equations in the displacements reduces to

$$\mu\nabla^2\underset{\sim}{u} + (\lambda+\mu)\text{grad div}\underset{\sim}{u} + \underset{\sim}{\hat{X}} = 0. \tag{1.6}$$

We have introduced here fictitious body forces

$$\hat{X}_i = -\sigma^0_{ji,j}, \tag{1.7}$$

where

$$\sigma^0_{ji} = 2\mu\varepsilon^0_{ij} + \lambda\delta_{ij}\varepsilon^0_{kk}. \tag{1.8}$$

Equations (1.6) should be completed by boundary conditions which may be given in terms of the displacement or of surface forces on the surface A bounding the body.

Solving the differential equations (1.6) we obtain the displacements $\underset{\sim}{u}$. Equation (1.4) serve then for the determination of the strain ε_{ij} and (1.3) makes it possible to calculate the stress σ_{ij}.

Equations (1.6) are particularly simple if we are considering thermal distortions, namely $\varepsilon_{ij} = \alpha_t \delta_{ij} T$. They then take the form

$$\mu\nabla^2\underset{\sim}{u} + (\lambda+\mu)\text{grad div}\underset{\sim}{u} = \gamma\text{grad}T, \quad \gamma = (3\lambda+2\mu)\alpha_t. \tag{1.6'}$$

The second method for determining the state of stress due to the action of distortions is the following:-

the components of the strain tensor satisfy the compatibility conditions

$$\varepsilon_{ij,kl}+\varepsilon_{kl,ij}-\varepsilon_{jl,ik}-\varepsilon_{ik,jl} = 0, \quad i,j,k,l = 1,2,3. \tag{1.9}$$

In view of relations (1.1) we have [101]

$$\varepsilon'_{ij,kl}+\varepsilon'_{kl,ij}-\varepsilon'_{jl,ik}-\varepsilon'_{ik,jl} = -(\varepsilon^0_{ij,kl}+\varepsilon^0_{kl,ij}-\varepsilon^0_{jl,ik}-\varepsilon^0_{ik,jl}). \tag{1.9'}$$

The right side of the equation (1.9') is known and constitutes the distortion terms. Writing the elastic strain ε'_{ij} in terms of stress σ_{ij} by means of relations (1.2) we obtain the compatability equations in terms of the stresses. Making use of the equations of equilibrium (1.5) we arrive at the Beltrami-Michell equations

$$\nabla^2\sigma_{ij} + \frac{1}{1+\nu}\sigma_{kk,ij} = -\mu(\Gamma_{ij}+\Gamma_{ji} + \frac{\nu}{1-\nu}\delta_{ij}\Gamma_{kk}), \tag{1.10}$$

$$\Gamma_{ij} = \nabla^2\varepsilon^0_{ij}+\varepsilon^0_{kk,ij}-\varepsilon^0_{jk,ik}-\varepsilon^0_{ik,jk}, \quad \nu = \frac{\lambda}{2(\lambda+\mu)}.$$

In the particular case of thermal distortion $\varepsilon^0_{ij} = \alpha_t\delta_{ij}T$, we have [101]

$$\nabla^2\sigma_{ij} + \frac{1}{1+\nu}\sigma_{kk,ij} = -\frac{\alpha_t E}{1+\nu}(T_{,ij} + \frac{1+\nu}{1-\nu}\delta_{ij}T_{,kk}). \tag{1.11}$$

Consider now the differential equations in displacements (1.6). Applying the divergence operator to equation (1.6) we have

$$\nabla^2 \mathrm{div}\,\underset{\sim}{u} = -\frac{1}{\lambda+2\mu}\mathrm{div}\,\hat{\underset{\sim}{X}}. \tag{1.12}$$

Next we apply the operator ∇^2 to equation (1.6) and make use of the relation (1.12) to arrive at the following equation containing only the displacement vector

$$\nabla^2\nabla^2\underset{\sim}{u} = -\frac{1}{\mu}\left(\nabla^2\hat{\underset{\sim}{X}} - \frac{\lambda+\mu}{\lambda+2\mu}\mathrm{grad}\ \mathrm{div}\,\hat{\underset{\sim}{X}}\right). \tag{1.13}$$

Equations (1.13) are very useful in determining the function $\underset{\sim}{u}$ due to the action of distortions in an infinite elastic space.

We now consider two particular examples.

1. First, we consider the distortion $\varepsilon^0_{ij} = \delta_{ij}e^0(\underset{\sim}{x})$. From (1.7) we have

$$\hat{X}_i = -(2\mu+3\lambda)e^0_{,i}. \tag{1.14}$$

Observe that in this case equation (1.13) is reduced to the equation

$$\nabla^2 u_i = me^0_{,i}, \quad m = \frac{3\lambda+2\mu}{\lambda+2\mu}. \tag{1.15}$$

Introducing the potential ϕ, we transform equation (1.15) to the form

$$\nabla^2\phi = me^0(\underset{\sim}{x}), \qquad u_i = \phi_{,i}. \tag{1.16}$$

The solution of the latter equation is

$$\phi(\underset{\sim}{x}) = -\frac{m}{4\pi}\int_V \frac{e^0(\underset{\sim}{\xi})dV(\underset{\sim}{\xi})}{R(\underset{\sim}{x},\underset{\sim}{\xi})}, \quad R = \left[(x_i-\xi_i)(x_i-\xi_i)\right]^{\frac{1}{2}}. \tag{1.17}$$

In the particular case of a "centre of dilatation" $e^0(\underset{\sim}{x}) = \delta(\underset{\sim}{x})$ we obtain

$$\phi(\underset{\sim}{x}) = -\frac{m}{4\pi R(\underset{\sim}{x},0)}, \qquad u_i = \phi_{,i}. \tag{1.18}$$

2. We now consider plane strain. Assume that all sources and unknown functions depend on the variable x_1, x_2.

In this particular case we obtain only one equation from (1.9'):

$$\partial_2^2\varepsilon'_{11} + \partial_1^2\varepsilon'_{22} - 2\partial_1\partial_2\varepsilon'_{12} = A \tag{1.19}$$

where

$$A = -(\partial_2^2\varepsilon_{11}^0 + \partial_1^2\varepsilon_{22}^0 + \partial_1\partial_2(\varepsilon_{12}^0 + \varepsilon_{21}^0)).$$

Replacing the strains ε'_{ij} by the stresses σ_{ij} and making use of the relations (1.2) we obtain the equation

$$\partial_2^2\sigma_{11} + \partial_1^2\sigma_{22} - \frac{\lambda}{2(\lambda+\mu)}\nabla_1^2(\sigma_{11}+\sigma_{22}) - 2\partial_1\partial_2\sigma_{12} = 2\mu A. \tag{1.20}$$

The compatability equation (1.20) should be completed by the equilibrium equations

$$\partial_1\sigma_{11}+\partial_2\sigma_{21} = 0, \quad \partial_1\sigma_{12}+\partial_2\sigma_{22} = 0. \tag{1.21}$$

If we express the stress by the Airy function

$$\sigma_{\alpha\beta} = (\nabla_1^2\delta_{\alpha\beta} - \partial_\alpha\partial_\beta)F, \quad \alpha,\beta = 1,2, \tag{1.22}$$

then the equilibrium equations are identically satisfied and the equation (1.21) is reduced to the simple differential equation

$$\nabla_1^2\nabla_1^2 F = \frac{4\mu(\lambda+\mu)}{\lambda+2\mu}A, \qquad \nabla_1^2 = \partial_1^2+\partial_2^2. \tag{1.23}$$

2. Basic relations and equations of thermoelasticity

A change in temperature of a body results in a state of stress and strain in it. The temperature T represents the increment of the temperature from the initial stress-free state when $T = 0$. We assume that the change of temperature is small and therefore has no influence on the mechanical and thermal properties of the body.

In the linear theory of elasticity the strain tensor ε_{ij} is connected with the displacement vector by the relations

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}). \tag{2.1}$$

The components of the strain tensor cannot be arbitrary; they should satisfy the following six relations - the so-called compatability conditions [101] -

$$\varepsilon_{ij,kl}+\varepsilon_{kl,ij}-\varepsilon_{jl,ik}-\varepsilon_{ik,jl} = 0, \qquad i,j,k,l = 1,2,3. \tag{2.2}$$

In the classical linear theory of elasticity the components of the strain tensor are functions of the components of the stress tensor and of the components of the strain tensor due to the temperature field

$$\varepsilon_{ij} = \varepsilon^{0}_{ij} + \varepsilon'_{ij}. \tag{2.3}$$

Here ε^{0}_{ij} denotes the deformation of an elementary parallelpiped due to the increment of temperature from zero to T under the assumption that the sides of the parallelpiped are free of tractions

$$\varepsilon^{0}_{ij} = \alpha_t\delta_{ij}T, \qquad i,j = 1,2,3. \tag{2.4}$$

The relation (2.4) represents a property of an isotropic body, in which a change of temperature results in no change of the shear angles, the only result being a change of volume of the elementary parallelpiped. In the relation (2.4) α_t denotes the coefficient of linear thermal expansion.

The strain tensor ε'_{ij} is expressed in terms of the stresses by the relations

$$\varepsilon'_{ij} = 2\mu'\sigma_{ij} + \lambda'\delta_{ij}\sigma_{kk} \tag{2.5}$$

where

$$2\mu' = \frac{1}{2\mu}, \qquad \lambda' = -\frac{\lambda}{2\mu(3\lambda+2\mu)}.$$

Here μ, λ denotes the Lamé material constants.

Introducing (2.5) into (2.3) and solving the system of relations for the stresses, we have

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + (\lambda\varepsilon_{kk} - \gamma T)\delta_{ij}, \quad \gamma = (3\lambda+2\mu)\alpha_t. \tag{2.6}$$

If in the equations of motion

$$\sigma_{ji,j} + X_i = \rho\ddot{u}_i, \qquad i,j = 1,2,3. \tag{2.7}$$

we express the stresses in terms of the strains and the latter by the displacements (making use of the relations (2.6) and (2.1)) we obtain a system of three equations in which the unknown functions are the components of the displacement vector and temperature [101]

$$\mu\nabla^2\underset{\sim}{u} + (\lambda+\mu)\text{grad div}\underset{\sim}{u} + \underset{\sim}{X} = \rho\ddot{\underset{\sim}{u}} + \gamma\text{grad}T, \tag{2.8}$$

or

$$\mu u_{i,jj} + (\lambda+\mu)u_{j,ji} + X_i = \rho\ddot{u}_i + \gamma T_{,i}.$$

From thermodynamical considerations we obtain the generalized equation of heat conduction |6|

$$\nabla^2 T - \frac{1}{\kappa}\dot{T} - \eta\text{div}\dot{\underset{\sim}{u}} = -\frac{W}{k}. \tag{2.9}$$

Here k denotes the coefficient of heat conduction, W denotes the quantity of heat generated in unit volume and unit time; $\kappa = {}^{k}/c_\varepsilon$, where c_ε is the specific heat, and $\eta = \frac{T_0\gamma}{k}$. The equations (2.8) and (2.9) are coupled. It is evident that the coupling of the temperature and strain field is due to the term $\eta\text{div}\dot{\underset{\sim}{u}}$.

The influence of the coupling of strain and temperature is appreciable only in the dynamic problem.

Consequently as a first approximation in the case of quasi-static and dynamic problems we need only consider the solution of the uncoupled problem.

In subsequent considerations we shall neglect the influence of the body forces, attention being devoted mainly to the stresses due to the temperature field. We shall consider the system of equations

$$\mu u_{i,jj} + (\lambda+\mu)u_{j,ji} = \rho\ddot{u}_i + \gamma T_{,i}, \tag{2.10}$$

$$\nabla^2 T - \frac{1}{\kappa}\dot{T} = -\frac{W}{k}, \tag{2.11}$$

with the boundary conditions

$$\sigma_{ji}n_j = 0, \quad \underset{\sim}{x} \in A, \quad t > 0 \tag{2.12}$$

for a body free from tractions on its boundary surface A and

$$u_i = U_i(\underset{\sim}{x},t), \tag{2.13}$$

for a body with kinematic boundary conditions on the whole surface A.

We now have to take into consideration the initial conditions

$$u_i(\underset{\sim}{x},0) = f_i(\underset{\sim}{x}), \ \dot{u}_i(\underset{\sim}{x},0) = g_i(\underset{\sim}{x}), \ \ \underset{\sim}{x} \in V, \ \ t = 0. \tag{2.14}$$

The temperature field is completely determined by equation (2.11) and the appropriate conditions, i.e. the boundary condition for $t > 0$ and the initial condition (for $t = 0$).

The boundary condition

$$\frac{\partial T}{\partial n} + \alpha T = \beta, \ \ \underset{\sim}{x} \in A, \ t > 0, \ \ \alpha, \ \beta \text{ are constants}, \tag{2.15}$$

represents free heat exchange on the surface A. If the body is thermally insulated, then $\frac{\partial T}{\partial n} = 0$ on A. If the temperature T is given on A, we have $T = h(\underset{\sim}{x},t)$, $\underset{\sim}{x} \in A$, $t > 0$.

The initial condition determines the distribution of temperature at the instant $t = 0$; this temperature in general is given as a function of position

$$T(\underset{\sim}{x},0) = s(\underset{\sim}{x}), \ \ \underset{\sim}{x} \in V, \ \ t = 0. \tag{2.16}$$

We shall represent the solution of the system of equations (2.10) for the static and quasi-static state

$$\mu u_{i,jj} + (\lambda+\mu)u_{j,ji} = \gamma T_{,i}, \tag{2.17}$$

in the form of the sum of two solutions - the particular solution of the non-homogeneous system and the general solution of the homogeneous system. The particular solution of system (2.17) can be simply derived by means of the method represented by J.N. Goodier [38] who introduced the so-called thermoelastic displacement potential Φ according to the relation

$$u_i = \Phi_{,i}. \tag{2.18}$$

Introducing (2.18) into (2.17) and integrating the above equations with respect to x_i, we arrive at the Poisson equation

$$\Phi_{,kk} = mT, \quad m = \gamma/\lambda+2\mu. \tag{2.19}$$

The solution of equation (2.19) yields the function Φ, and the strain and stresses can be calculated by means of the following relations

$$\varepsilon_{ij} = \Phi_{,ij}, \quad \sigma_{ij} = 2\mu(\Phi_{,ij}-\delta_{ij}\Phi_{,kk}). \tag{2.20}$$

For an unbounded body the functions $u_i = \Phi_{,i}$ represent the final solution of the system of equations (2.17).

For a bounded body the function Φ satisfies only a part of the boundary conditions. Therefore the incomplete solution $\bar{u}_i = \Phi_{,i}$ must be completed by a solution $\bar{\bar{u}}_i$ of the system of homogeneous equations

$$\mu\bar{\bar{u}}_{i,jj} + (\lambda+\mu)\bar{\bar{u}}_{j,ji} = 0. \tag{2.21}$$

The functions $\bar{\bar{u}}_i$ must so be chosen that the final solution $u_i = \bar{u}_i + \bar{\bar{u}}_i$ satisfies all the boundary conditions of the problem.

The system of equations (2.21) can, for instance, be solved by means of the Galerkin functions. Introducing the Galerkin vector $\underset{\sim}{\phi}(\underset{\sim}{x})$, we assume the representation of $\underset{\sim}{u}$ in the form [36]

$$\bar{\bar{\underset{\sim}{u}}} = \frac{\lambda+2\mu}{\mu}\left(\nabla^2\underset{\sim}{\phi} - \frac{\lambda+\mu}{\lambda+2\mu}\text{grad div}\underset{\sim}{\phi}\right). \tag{2.22}$$

Substituting from (2.22) into the system of equations (2.21) we obtain a system of three biharmonic equations

$$\nabla^2\nabla^2\underset{\sim}{\phi} = 0. \tag{2.23}$$

In numerous problems of thermoelasticity, particularly in the case of stress boundary conditions, it is more convenient to use the stress equations of equilibrium, representing a generalization of the equation of E. Beltrami and I.H. Michell to the case of thermal stresses.

Introducing the strain tensor ε_{ij} given by formulae (2.3) and (2.5)

$$\varepsilon_{ij} = \alpha_t\delta_{ij}T + 2\mu'\sigma_{ij} + \lambda'\delta_{ij}\sigma_{kk} \tag{2.24}$$

into the compatibility equations (2.2) and making use of the equilibrium equations, we obtain the system of six equations [101]

$$\sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{kk,ij} + \frac{\alpha_t E}{1+\nu}\left(T_{,ij} + \frac{1+\nu}{1-\nu}\delta_{ij}T_{,kk}\right) = 0, \qquad i,j = 1,2,3 \tag{2.25}$$

where

$$\nu = \frac{\lambda}{2(\lambda+\mu)}, \quad E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}.$$

Contraction of the above equations yields

$$\sigma_{kk,jj} + \frac{2\alpha_t E}{1-\nu}T_{,kk} = 0. \tag{2.26}$$

In the particular case of a stationary temperature field with no heat sources acting inside the body, equations (2.25) are significantly simplified. Since in this case $T_{,kk} = 0$, then according to the relation (2.26) we have $\sigma_{kk,jj} = 0$. Hence

$$\sigma_{ij,kk} + \frac{1}{1+\nu}(\sigma_{kk} + \alpha_t ET)_{,ij} = 0. \tag{2.27}$$

Any non-stationary problem of thermoelasticity is basically a problem of dynamic elasticity. Only if the changes in temperature are slow can the problem be regarded as quasi-static. If however there exist sudden changes of temperature (for instance a sudden heating or cooling of the surface of the body) the inertia terms cannot be neglected. We then have to investigate the equations of motion (2.10).

We rewrite the system of equations (2.10) in the vector form

$$\mu\nabla^2\underset{\sim}{u} + (\lambda+\mu)\text{grad div}\underset{\sim}{u} - \rho\ddot{\underset{\sim}{u}} = \gamma\text{grad}T. \qquad (2.28)$$

The displacement $\underset{\sim}{u}$ may be represented by

$$\underset{\sim}{u} = \text{grad}\phi + \text{curl}\underset{\sim}{\Psi};\ \text{div}\underset{\sim}{\Psi} = 0. \qquad (2.29)$$

Substituting from (2.29) into (2.28), we conclude that the system of equations (2.28) is satisfied if

$$\square_1^2\phi = m\theta, \quad \square_2^2\underset{\sim}{\Psi} = 0 \qquad (2.30)$$

where

$$\square_\alpha^2 = \nabla^2 - \frac{1}{c_\alpha^2}\partial_t^2, \quad \alpha = 1,2;\ c_1 = \left(\frac{\lambda+2\mu}{\rho}\right)^{\frac{1}{2}},\ c_2 = \left(\frac{\mu}{\rho}\right)^{\frac{1}{2}}.$$

Here c_1 denotes the velocity of propagation of an irrotational wave resulting only in a change of volume. The velocity c_1 is associated with equation $(2.30)_1$ which determines the potential (irrotational) displacement field. The quantity c_2 is the velocity of propagation of rotational motion in which elements of the body suffer equivoluminal changes. A temperature field in a bounded body results in both types of waves.

Eliminating the temperature from the heat conduction equation and equation $(2.30)_1$, we arrive at the system of wave equations

$$\square_1^2\square_3^2\phi = -\frac{mW}{k}, \quad \square_2^2\underset{\sim}{\Psi} = 0, \qquad (2.31)$$

where

$$\square_3^2 = \nabla^2 - \frac{1}{\kappa}\partial_t.$$

If the function ϕ is known the displacements and stresses in the infinite space can easily be found. From the strain-stress law (2.6) and from the equation (2.30) we obtain

$$\sigma_{ij} = 2\mu(\phi_{,ij} - \delta_{ij}\phi_{,kk}) + \rho\delta_{ij}\ddot{\phi}. \qquad (2.32)$$

If the body is bounded, we have to solve the system of equations (2.30) with the appropriate boundary and initial conditions. But the incomplete solution $\overline{u}_i = \phi_{,i} + \epsilon_{ijk}\Psi_{k,j}$ must be completed by the solution $\overline{\overline{u}}_i$ of the system of homogeneous equations

$$\mu\nabla\overline{\overline{u}} + (\lambda+\mu)\mathrm{grad}\,\mathrm{div}\overline{\overline{u}} - \rho\ddot{\overline{\overline{u}}} = 0. \tag{2.33}$$

The solution $\overline{\overline{u}}_i$ can be represented by means of the M. Iacovache vector ϕ [48]

$$\overline{\overline{u}} = \frac{\lambda+2\mu}{\mu}\left(\Box_1^2\phi - \frac{\lambda+\mu}{\lambda+2\mu}\mathrm{grad}\,\mathrm{div}\phi\right). \tag{2.34}$$

The function ϕ satisfies the equation

$$\Box_1^2\Box_2^2\phi = 0. \tag{2.35}$$

In some cases it is simpler to determine the stresses from the stress equation of motion. The equations given below were derived by J. Ignaczak [49] and they constitute an extension of the equations of E. Beltrami and J.H. Michell to dynamic problems of thermoelasticity.

We obtain the system of equations

$$\Box_2^2\sigma_{ij} + \frac{2(\lambda+\mu)}{3\lambda+2\mu}\sigma_{kk,ij} + \left(\frac{1}{c_2^2} - \frac{1}{c_1^2}\right)\frac{\lambda\delta_{ij}}{2\mu+3\lambda}\ddot{\sigma}_{kk} +$$

$$+ 2\mu\alpha_t\left(T_{,ij} + \frac{3\lambda+2\mu}{\lambda+2\mu}\delta_{ij}T_{,kk}\right) - \frac{5\lambda+4\mu}{\lambda+2\mu}\alpha_t\rho\delta_{ij}\ddot{T} = 0.$$

Applying the operator $\Box_1^2$ to equation (2.36) we obtain the interesting result

$$\Box_1^2\Box_2^2\sigma_{ij} = m\,\Box_1^2\left[2\mu(T_{,ij}-\delta_{ij}T_{,kk}) + \rho\delta_{ij}\ddot{T}\right].$$

We now consider the coupled equations of thermoelasticity (2.8) and (2.9).

Assuming that

$$\begin{aligned} u &= \mathrm{grad}\phi + \mathrm{curl}\Psi, \quad \mathrm{div}\Psi = 0 \\ X &= \rho(\mathrm{grad} \quad + \mathrm{curl}\chi), \quad \mathrm{div}\chi = 0 \end{aligned} \tag{2.37}$$

we obtain the following system of wave-equations

$$\square_1^2\phi = mT - \frac{1}{c_1^2}\vartheta,$$
$$\square_3^2 T - \eta\nabla^2\dot{\phi} = -\frac{W}{k}, \tag{2.38}$$

and

$$\square_2^2\underset{\sim}{\psi} = -\frac{1}{c_2^2}\underset{\sim}{\chi}. \tag{2.39}$$

The first two equations are coupled.

Now we assume that the initial conditions are homogeneous.

We now confine ourselves to the case of an infinite space. If the body forces are absent ($\vartheta = 0$, $\underset{\sim}{\chi} = 0$) and the state of stress is produced by the action of heat sources, there arise only longitudinal waves, for $\underset{\sim}{\chi} = 0$. The function ϕ is to be determined from the wave equation [127]

$$(\square_1^2\square_3^2 - \frac{\varepsilon}{\kappa}\alpha_t\nabla^2)\phi = -\frac{mW}{k}, \quad \varepsilon = \eta m\kappa, \tag{2.40}$$

and the temperature is determined from equation $(2.38)_2$. The displacements are given by the formula $u_i = \phi_{,i}$ and the stresses by the relation

$$\sigma_{ij} = 2\mu(\phi_{,ij} - \delta_{ij}\phi_{,kk}) + \rho\delta_{ij}\ddot{\phi}. \tag{2.41}$$

Suppose that in the infinite space there are no heat sources ($W = 0$) and that the body forces can be derived from a potential ($\chi_i = \rho\vartheta_{,i}$, $\underset{\sim}{\chi} = 0$).

In this case also, only the longitudinal wave occurs since $\psi = 0$. The function ϕ is obtained from the equation

$$(\square_1^2\square_3^2 - \frac{\varepsilon}{\kappa}\alpha_t\nabla^2)\phi = -\frac{1}{c_1^2}\square_3^2\vartheta , \tag{2.42}$$

and the temperature from the equation $(2.38)_2$. The displacements u_i are given by formulae $u_i = \phi_{,i}$ and the stresses by the relations

$$\sigma_{ij} = 2\mu\varepsilon_{ij}+(\lambda\varepsilon_{kk}-\gamma T)\delta_{ij} = 2\mu(\phi_{,ij}-\delta_{ij}\phi_{,kk})+\rho\delta_{ij}(\ddot{\Phi}-\vartheta). \tag{2.43}$$

Finally if the body forces are of rotational nature ($\vartheta = 0$, $\underset{\sim}{X} = \rho\,\mathrm{curl}\chi$) only transverse waves are present in the body ($\phi = 0$) which result in no temperature field ($T = 0$).

3. Thermal inclusions. Nucleus of thermoelastic strain

Discontinuous temperature fields are frequently encountered in engineering practice. The simplest example is the case in which a part of the body is heated to a constant temperature $T^{(i)}$, the remaining part having a temperature $T^{(e)}$; this case occurs when we are dealing with a thermal inclusion.

We know that in an infinite space the displacement equations of thermoelasticity have solution $u_i = \Phi_{,i}$, where

$$\nabla^2\Phi = mT, \qquad m = \frac{\gamma}{\lambda+2\mu}, \tag{3.1}$$

and which can therefore be represented by the Poisson integral

$$\Phi(\underset{\sim}{x}) = -\frac{m}{4\pi}\int_{v}\frac{T(\underset{\sim}{\xi})dV(\underset{\sim}{\xi})}{R(\underset{\sim}{x},\underset{\sim}{\xi})}, \quad R = \left[(x_i-\xi_i)(x_i-\xi_i)\right]^{\frac{1}{2}}. \tag{3.2}$$

The solution (3.2) is valid for a continuous temperature field, assuming the existence of the first derivatives of the function T, which appear in the right-hand side of the displacement equation of thermoelasticity.

It is proved in the potential theory that if it is assumed that the function $T(\underset{\sim}{\xi})$ is integrable and bounded, then the integral in formula (3.2) is also a solution of equation (3.1). Such being the case the function ϕ has the following properties.

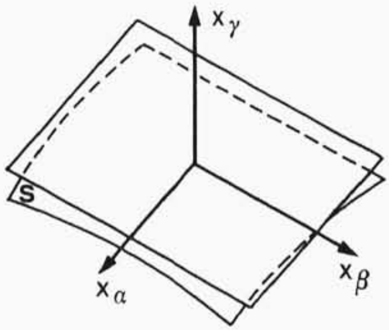

Fig. 3.1

a) The first derivatives $\frac{\partial\phi}{\partial x_i}$ of ϕ are continuous also in regions where T has discontinuities.

b) The second derivatives of the function ϕ are continuous in the whole region, except the derivative $\phi_{,\gamma\gamma}$ which on the discontinuity surface S has the discontinuity (Fig. 3.1).

$$\phi^{(i)}_{,\gamma\gamma} - \phi^{(e)}_{,\gamma\gamma} = m(T^{(i)} - T^{(e)}). \tag{3.3}$$

According to the relation $u_i = \phi_{,i}$, the first property implies that the components of the displacement vector are continuous. It follows further from the second property that the tangential-stresses $\sigma_{\gamma\alpha}$, $\sigma_{\gamma\beta}$ are continuous. Similarly the stress $\sigma_{\gamma\gamma}$ is a continuous function. In fact, from formula $(2.20)_2$ we obtain

$$\sigma^{(i)}_{\gamma\gamma} - \sigma^{(e)}_{\gamma\gamma} = 2\mu\left[\phi^{(i)}_{,\gamma\gamma} - \phi^{(e)}_{,\gamma\gamma} - \nabla^2(\phi^{(i)} - \phi^{(e)}\right].$$

Taking into account the relation (3.3) and bearing in mind that

$$\nabla^2(\phi^{(i)} - \phi^{(e)}) = m(T^{(i)} - T^{(e)}) \tag{3.4}$$

we find that on passing through the surface S we have $\sigma^{(i)}_{\gamma\gamma} = \sigma^{(e)}_{\gamma\gamma}$

On the other hand the stresses $\sigma_{\alpha\alpha}$, $\sigma_{\beta\beta}$ suffer a discontinuity on passing through S, the value of which is $-2\mu m(T^{(i)}-T^{(e)})$.

In fact, we have from equation $(2.20)_2$

$$\sigma^{(i)}_{\alpha\alpha}-\sigma^{(e)}_{\alpha\alpha} = 2\mu\left[\phi^{(i)}_{,\alpha\alpha}-\phi^{(e)}_{,\alpha\alpha}-\nabla^2(\phi^{(i)}-\phi^{(e)})\right]. \tag{3.5}$$

In view of the continuity of the function $\phi_{,\alpha\alpha}(\phi^{(i)}_{,\alpha\alpha} = \phi^{(e)}_{,\alpha\alpha})$ expression (3.5) is reduced to

$$\sigma^{(i)}_{\alpha\alpha}-\sigma^{(e)}_{\alpha\alpha} = -2\mu m(T^{(i)}-T^{(e)}), \tag{3.6}$$

by virtue of (3.4). In an analogous way we obtain

$$\sigma^{(i)}_{\beta\beta}-\sigma^{(e)}_{\beta\beta} = -2\mu m(T^{(i)}-T^{(e)}). \tag{3.7}$$

Let us return to the solution (3.2) of equation (3.1) for an infinite space. The Poisson integral can be represented in the form

$$\phi(\underset{\sim}{x}) = \int_V T(\underset{\sim}{\xi})\phi^{*}(\underset{\sim}{x},\underset{\sim}{\xi})dV(\underset{\sim}{x}). \tag{3.8}$$

The function

$$\phi^{*}(\underset{\sim}{x},\underset{\sim}{\xi}) = -\frac{m}{4\pi}\frac{1}{R(\underset{\sim}{x},\underset{\sim}{\xi})} \tag{3.9}$$

is the Green's function of equation (3.1) in an infinite body. It satisfies the differential equation

$$\nabla^2\phi^{*} = m\delta(\underset{\sim}{x}-\underset{\sim}{\xi}). \tag{3.10}$$

By means of the function ϕ^{*} we can express the displacements and stresses in an integral form; we have

$$u_i(\underset{\sim}{x}) = \int_V T(\underset{\sim}{\xi})\frac{\partial}{\partial x_i}\phi^{*}(\underset{\sim}{x},\underset{\sim}{\xi})dV(\underset{\sim}{\xi}) \tag{3.11}$$

and

$$\sigma_{ij}(\underset{\sim}{x}) = \int_V T(\xi)\sigma^{*}_{ij}(\underset{\sim}{x},\underset{\sim}{\xi})dV(\underset{\sim}{\xi}), \tag{3.12}$$

where

$$\overset{*}{\sigma}_{ij}(\underset{\sim}{x},\underset{\sim}{\xi}) = 2\mu(\overset{*}{\phi}_{,ij} - \delta_{ij}\overset{*}{\phi}_{,kk}).$$

The functions $\overset{*}{u}_i$, $\overset{*}{\sigma}_{ij}$ are the Green's functions for the displacements and stresses respectively.

In many cases it is more convenient to make use of the direct solution of the equation for the thermoelastic displacement potential (3.1).

We now consider a thermal inclusion having the form of a semi-infinite cylinder. This problem was dealt with by N.O. Myklestad |72|; we shall however use a different method - the integral transform method.

The temperature field can be expressed in terms of the Heaviside function as follows

$$T = H(a-r)H(z), \quad r = (x_1^2+x_2^2)^{\frac{1}{2}}, \quad z = x_3. \tag{3.13}$$

This means that inside the region of the semi-infinite circular cylinder of radius a the temperature is $T = 1$, and outside this region the temperature is zero. We now represent the field (3.13) as the sum of two fields - the first anti-symmetric, the second - symmetric with respect to the plane $z = 0$. Hence

$$T = T^{(1)}+T^{(2)}$$

where

$$T^{(1)} = \tfrac{1}{2}H(a-r)\left[H(z)-H(-z)\right],$$

$$T^{(2)} = \tfrac{1}{2}H(a-r)\left[H(z)+H(-z)\right].$$

Consider first the action of the anti-symmetric temperature field $T^{(1)}$ which can be represented by the integral*

* $$\int_0^a rJ_0(\alpha r)\mathrm{d}r = \frac{a}{\alpha}J_1(\alpha a).$$

$$T^{(1)} = \frac{a}{\pi}\int_0^\infty\int_0^\infty \frac{\sin\gamma z}{\gamma} J_1(\alpha a) J_0(\alpha r)\,d\alpha\, d\gamma. \tag{3.14}$$

The thermoelastic displacement potential $\phi^{(1)}$ is determined by solving the equation (3.1) by means of the Hankel and Fourier sine transforms

$$\phi^{(1)} = -\frac{ma}{\pi}\int_0^\infty\int_0^\infty \frac{\gamma^{-1}\sin\gamma z}{\alpha^2+\gamma^2} J_1(\alpha a) J_0(\alpha r)\,d\alpha\, d\gamma \tag{3.15'}$$

or

$$\phi^{(1)} = \frac{ma}{2}\int_0^\infty (e^{-\alpha z}-1)\frac{J_1(\alpha a)}{\alpha^2} J_0(\alpha r)\,d\alpha, \quad z > 0. \tag{3.15''}$$

The stresses $\sigma_{ij}^{(1)}$ corresponding to the temperature field are given by the formulae $(2.20)_2$, but in cylindrical coordinates

$$\sigma_{rr}^{(1)} = -\mu ma\int_0^\infty J_1(\alpha a)\{ e^{-\alpha z}\left[J_0(\alpha r) - \frac{J_1(\alpha r)}{\alpha r}\right] + \frac{J_1(\alpha r)}{\alpha r}\}\,d\alpha,$$

$$\sigma_{\phi\phi}^{(1)} = -\mu ma\int_0^\infty J_1(\alpha a)\left[(e^{-\alpha z}-1)\frac{J_1(\alpha r)}{\alpha r} + J_0(\alpha r)\right]d\alpha, \tag{3.16}$$

$$\sigma_{zz}^{(1)} = \mu ma\int_0^\infty (e^{-\alpha z}-1) J_1(\alpha a) J_0(\alpha r)\,d\alpha,$$

$$\sigma_{rz}^{(1)} = \mu ma\int_0^\infty e^{-\alpha z} J_1(\alpha a) J_0(\alpha r)\,d\alpha.$$

The above formulae hold for $z > 0$. For $z < 0$, z has to be replaced by $-z$ and the signs changed. Let us observe that the temperature field $T^{(2)}$ yields the temperature $T^{(2)} = \frac{1}{2}$ inside the infinite circular cylinder of radius a and $T^{(2)} = 0$ outside this region. We obtain here a plane state of strain in which the displacements, strains and stresses are independent of the variable z. The stresses $\sigma_{ij}^{(2)}$ are determined from the formulae for the stresses $\sigma_{ij}^{(1)}$ for $z > 0$, letting $z \to \infty$ in (3.16). Taking into account that

$$\int_0^\infty \frac{J_1(\alpha a)}{\alpha r} J_1(\alpha r)\mathrm{d}\alpha = \frac{1}{2a}\left[H(a-r) + \left(\frac{a}{r}\right)^2 H(r-a)\right],$$

$$\int_0^\infty J_0(\alpha r)J_1(\alpha a)\mathrm{d}\alpha = \frac{1}{a}\, H(a-r),$$

we obtain

$$\begin{aligned}
\sigma_{rr}^{(2)} &= -\frac{\mu m}{a}\left[H(a-r) + \frac{a^2}{r^2}H(r-a)\right],\\
\sigma_{\phi\phi}^{(2)} &= -\frac{\mu m}{a}\left[H(a-r) - \frac{a^2}{r^2}H(r-a)\right],\\
\sigma_{zz}^{(2)} &= -\mu m H(a-r),\ \sigma_{zr}^{(2)} = 0.
\end{aligned} \tag{3.17}$$

The stresses $\sigma_{ij}^{(2)}$ are constant inside the cylinder and for $r > a$ rapidly tend to zero. The stresses $\sigma_{\phi\phi}^{(2)}$, $\sigma_{zz}^{(2)}$ suffer a discontinuity on the surface $r = a$, and the stress $\sigma_{\phi\phi}^{(2)}$ changes sign on passing through it. (Cf. Fig. 3.2)).

Adding the stresses $\sigma_{ij}^{(1)}$ and $\sigma_{ij}^{(2)}$ we obtain the final stresses due to the action of the temperature field (3.13). The Hankel integrals appearing in the relations for the stresses can be represented by means of elliptic integrals and Legendre functions.

To this end we observe that [32]

$$\begin{aligned}
I_1(a,r,z) &= \int_0^\infty e^{-\alpha z}J_0(\alpha a)J_0(\alpha r)\mathrm{d}\alpha = (\pi^2 ar)^{-\frac{1}{2}}Q_{-\frac{1}{2}}\left(\frac{r^2+z^2+a^2}{2ar}\right),\\
I_2(a,r,z) &= \int_0^\infty e^{-\alpha z}J_1(\alpha a)J_1(\alpha r)\mathrm{d}\alpha = (\pi^2 ar)^{-\frac{1}{2}}Q_{\frac{1}{2}}\left(\frac{r^2+z^2+a^2}{2ar}\right),\\
I_3(a,r,z) &= \int_0^\infty e^{-\alpha z}J_1(\alpha a)J_0(\alpha r)\mathrm{d}\alpha = (\pi a)^{-1}\Big\{K'E(k,\theta)+(E'-K')F(k,\theta) -\\
&\qquad - z\left[(a+r)^2+z^2\right]^{-\frac{1}{2}}K'\Big\},
\end{aligned} \tag{3.18}$$

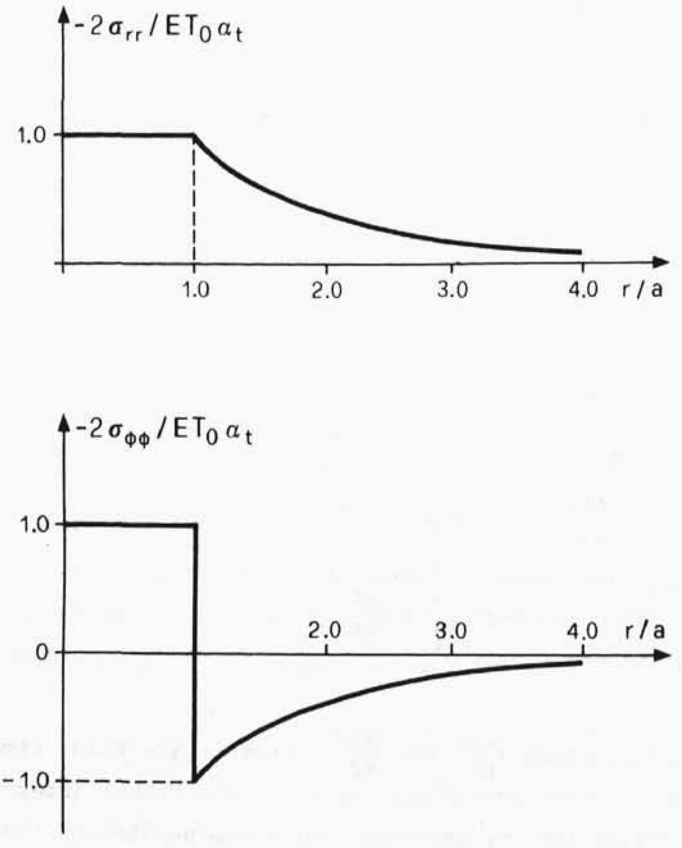

Fig. 3.2

where

$$I_1(a,r,z) = I_1(r,a,z), \; I_2(a,r,z) = I_2(r,a,z), \;\; I_3(a,r,z) \neq I_3(r,a,z).$$

The functions $Q_{\frac{1}{2}}(\frac{r^2+a^2+z^2}{2ar})$, $Q_{-\frac{1}{2}}(\frac{r^2+a^2+z^2}{2ar})$ are Legendre functions of the second kind, $F(k,\theta)$ and $E(k,\theta)$ denote the incomplete elliptic integrals of the first and second kind, respectively, with modulus $k = \left[(a-r)^2 + z^2\right]^{\frac{1}{2}} \left[(a+r)^2+z^2\right]^{-\frac{1}{2}}$ and argument $\theta = \sin^{-1}\{z\left[(a-r)^2+z^2\right]^{-\frac{1}{2}}\}$

where $0 < \theta < \pi$. Finally K' and E' denote the complete elliptic integrals of the first and second kind, respectively, referred to the complementary modulus $k' = (1-k)^{\frac{1}{2}}$. Adding the stresses $\sigma_{ij}^{(1)}$ and $\sigma_{ij}^{(2)}$, making use of formulae (3.18) and of the relation

$$\int_0^\infty e^{-\alpha z}\alpha^{-1}J_1(\alpha a)J_1(\alpha r)d\alpha = \tfrac{1}{2}\left[aI_3(r,a,z)+rI_3(a,r,z)-zI_2(a,r,z)\right] \qquad (3.19)$$

we obtain

$$\sigma_{rr} = -\mu ma\{I_3(a,r,z)- \frac{1}{2r}\left[aI_3(r,a,z) + rI_3(a,r,z)-zI_2(a,r,z)\right] + + \frac{1}{a}\left[H(a-r) + \frac{a^2}{r^2} H(r-a)\right]\}$$

$$\sigma_{\phi\phi} = -\mu ma\{\frac{1}{2r}\left[aI_3(r,a,z)+rI_3(a,r,z)-zI_2(a,r,z)\right] + + \frac{1}{a}\left[H(a-r) - \frac{a^2}{r^2}H(r-a)\right]\}, \qquad (3.20)$$

$$\sigma_{zz} = \mu ma\left[I_3(a,r,z)- \frac{2}{a}H(a-r)\right],$$

$$\sigma_{rz} = \mu maI_2(a,r,z).$$

For $z < 0$ we have

$$\sigma_{rr} = \mu ma\{I_3(a,r,z)- \frac{1}{2r}\left[aI_3(r,a,z)+ rI_3(a,r,z)+ zI_2(a,r,z)\right]\}.$$

$$\sigma_{\phi\phi} = \frac{\mu ma}{2r} \left[aI_3(r,a,z)+ rI_3(a,r,z)+ zI_2(a,r,z)\right], \qquad (3.21)$$

$$\sigma_{zz} = -\mu maI_3(a,r,z),$$

$$\sigma_{rz} = -\mu maI_2(a,r,z).$$

The above formulae take a particularly simple form for $r = 0$.

Thus for $z > 0$, we have

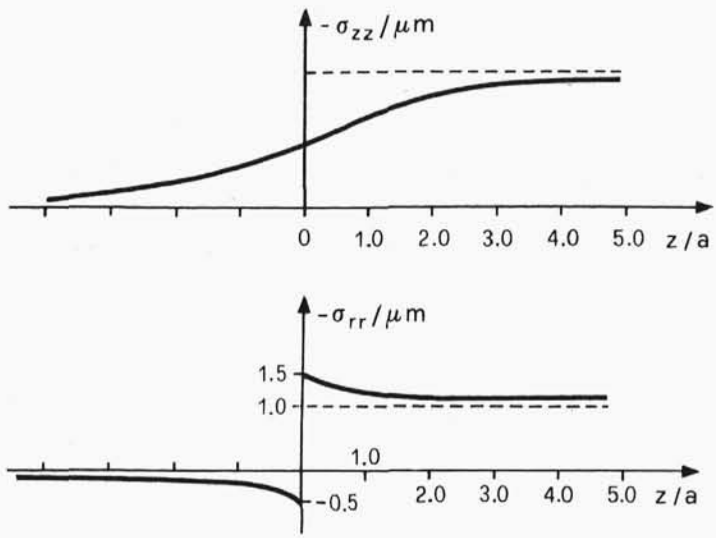

Fig. 3.3

$$\sigma_{rr} = \sigma_{\phi\phi} = -\frac{\mu m}{2}\left(3 - \frac{z}{\sqrt{z^2+a^2}}\right),$$

$$\sigma_{zz} = -\mu m\left(1 + \frac{z}{\sqrt{z^2+a^2}}\right), \quad \sigma_{rz} = 0; \tag{3.22}$$

and for $z < 0$

$$\sigma_{rr} = \sigma_{\phi\phi} = \frac{\mu m}{2}\left(1 + \frac{z}{\sqrt{z^2+a^2}}\right),$$

$$\sigma_{zz} = -\mu m\left(1 - \frac{z}{\sqrt{z^2+a^2}}\right), \quad \sigma_{rz} = 0. \tag{3.23}$$

The discontinuity in the stresses σ_{rr} and $\sigma_{\phi\phi}$ at the point $z = 0$ on the z-axis has the value $-2\mu m$. Fig. (3.3) shows the graphs of the stresses σ_{rr}, σ_{zz}, for $r = 0$.

4. Heat sources in an elastic space

In numerous cases of practical interest we encounter heat sources acting inside the elastic body. These sources may be point, line, surface and volume sources in an infinite elastic space. The Green's functions for a concentrated source can then be employed to construct more general solutions concerning sources distributed in a continuous or discontinuous manner in the space.

If we locate a heat source at the origin of the coordinate system, we have to solve the equations

$$\nabla^2 T = -\frac{Q}{\kappa}, \qquad \nabla^2\phi = mT, \qquad Q = \frac{W}{k}\kappa, \tag{4.1}$$

or the equation

$$\nabla^2\nabla^2\phi = -\frac{m}{\kappa}Q, \quad Q(\mathrm{x}) = Q_0\delta(\mathrm{x}). \tag{4.2}$$

The solution of (4.2) has the form

$$\phi = \frac{mQ_0}{8\pi\kappa}R + \text{const.} \qquad R = (x_1^2+x_2^2+x_3^2)^{\frac{1}{2}}. \tag{4.3}$$

If we now situate a concentrated heat source of unit intensity $Q_0 = 1$ at the point ξ, we have the following relations for the component of the stress tensor

$$\overset{*}{\sigma}_{ij} = 2\mu(\phi_{,ij}-\delta_{ij}\phi_{,kk}),$$

or

$$\overset{*}{\sigma}_{ij} = -\frac{2\mu A}{R}\left[\delta_{ij} + (x_i-\xi_i)(x_j-\xi_j)R^{-2}\right], \; i,j = 1,2,3, \tag{4.4}$$

where

$$R = \left[(x_i-\xi_i)(x_i-\xi_i)\right]^{\frac{1}{2}}, \qquad A = \frac{m}{8\pi\kappa}.$$

The functions (4.4) should be regarded as the Green's functions for the stresses in an infinite body. If the sources are distributed in a volume V,

$$\sigma_{ij}(\underset{\sim}{x}) = \int_V Q(\underset{\sim}{\xi})\sigma^{*}_{ij}(\underset{\sim}{\xi},\underset{\sim}{x})dV(\underset{\sim}{\xi}). \tag{4.5}$$

In many cases it is more expedient to determine the stresses by a direct integration of the differential equation (4.2).

Let us examine the action of heat sources distributed uniformly inside a circle of radius a. In this particular case we have

$$\nabla^2\nabla^2\phi = -\frac{m}{\kappa}Q_0 H(a-r)\delta(z), \tag{4.6}$$

where $H(\eta)$ is the Heaviside function. This is an axisymmetric problem.

We apply the Fourier cosine and Hankel transforms to the equation and obtain as a result

$$\phi = -\frac{maQ_0}{4\kappa}\int_0^\infty \frac{J_1(\alpha a)}{\alpha^3}J_0(\alpha r)(1+\alpha z)e^{-\alpha z}d\alpha, \quad z > 0. \tag{4.7}$$

This integral is divergent in the entire space. We can however separate the divergent and convergent parts, the divergent part having no influence on the gradient of the function.

The temperature field is represented by the formula

$$T = \frac{1}{m}\nabla^2\phi = \frac{Q_0 a}{2\kappa}\int_0^\infty \frac{e^{-\alpha z}}{\alpha}J_1(\alpha a)J_0(\alpha r)d\alpha. \tag{4.8}$$

Having found the function ϕ we readily determine the components of the stress tensor for $z > 0$:

$$\sigma_{rr} = B\int_0^\infty \alpha^{-1}J_1(\alpha a)e^{-\alpha z}\left[(\alpha z-1)J_0(\alpha r) - (1+\alpha z)\frac{J_1(\alpha r)}{\alpha r}\right]d\alpha,$$

$$\sigma_{\phi\phi} = B\int_0^\infty \alpha^{-1}J_1(\alpha a)e^{-\alpha z}\left[(1+\alpha z)\frac{J_1(\alpha r)}{\alpha r} - 2J_0(\alpha r)\right]d\alpha, \tag{4.9}$$

$$\sigma_{zz} = -B\int_0^\infty \alpha^{-1} J_1(\alpha a) J_0(\alpha r) e^{-\alpha z}(1+\alpha z)\,d\alpha,$$

$$\sigma_{rz} = -Bz\int_0^\infty J_1(\alpha a) J_1(\alpha r) e^{-\alpha z}\,d\alpha,\ \sigma_{r\phi} = 0,\ B = \frac{\mu m a Q_0}{2\kappa}.$$

The integrals occurring in the above expressions

$$\int_0^\infty \alpha^{\lambda-1} e^{-\alpha z} J_1(\alpha a) J_\nu(\alpha r)\,d\alpha, \qquad \nu = 0,1, \tag{4.10}$$

can be represented in terms of the functions I_1, I_2, I_3 of section 1.3, or by means of a series of hypergeometric functions [32].

Any non-stationary state of stress constitutes a dynamic problem. However, if the variation of the temperature field with time is small the acceleration may be neglected and the problem regarded as quasistatic.

Consider the system of equations

$$(\nabla^2 - \frac{1}{\kappa}\partial_t)T = -\frac{Q}{\kappa},\quad (\nabla^2 - \frac{1}{c_1^2}\partial_t^2)\phi = mT, \tag{4.11}$$

for the dynamic problem. Let us perform the Laplace transform on equation (4.11), assuming that the initial state is the natural state: thus

$$(\nabla^2 - \frac{p}{\kappa})\overline{T} = -\frac{\overline{Q}}{\kappa},\ (\nabla^2 - \frac{p^2}{c_1^2})\overline{\phi} = m\overline{T}, \tag{4.12}$$

where

$$(\overline{T},\overline{\phi}) = \int_0^\infty (T,\phi)e^{-pt}\,dt.$$

Eliminating the function $\overline{T}$, we find that $\overline{\phi}$ should satisfy the equation

$$(\nabla^2 - \frac{p}{\kappa})(\nabla^2 - p^2\sigma^2)\overline{\phi} = -\frac{m}{\kappa}\overline{Q}, \qquad \sigma^2 = 1/c_1^2. \tag{4.12'}$$

We now apply to equation (4.12') the exponential Fourier transform in accordance with the relations

$$\tilde{f}(\underset{\sim}{\xi},p) = (2\pi)^{-3/2}\int_{E_3} \bar{f}(\underset{\sim}{x},p)\exp(i\underset{\sim}{x}.\underset{\sim}{\xi})dV(\underset{\sim}{x}),$$

$$\bar{f}(\underset{\sim}{x},p) = (2\pi)^{-3/2}\int_{E_3} \tilde{f}(\underset{\sim}{\xi},p)\exp(-i\underset{\sim}{x}.\underset{\sim}{\xi})dW(\underset{\sim}{\xi}), \tag{4.13}$$

$$dV = dx_1 dx_2 dx_3, \qquad dW = d_{\xi_1} d_{\xi_2} d_{\xi_3}.$$

The solution of equation (4.12') is given by the integral

$$\bar{\phi}(\underset{\sim}{x},p) = -\frac{m}{\kappa}(2\pi)^{-3/2}\int_{E_3} \frac{\tilde{Q}(\underset{\sim}{\xi},p)\exp(-i\underset{\sim}{\xi}\underset{\sim}{x})dW(\underset{\sim}{\xi})}{(\underset{\sim}{\xi}.\underset{\sim}{\xi}+p/\kappa)(\underset{\sim}{\xi}.\underset{\sim}{\xi}+p^2\sigma^2)}, \tag{4.14'}$$

or

$$\phi(\underset{\sim}{x},p) = -\frac{m(2\pi)^{-3/2}}{p\sigma^2-p/\kappa}\{\frac{1}{\kappa}\int_{E_3} \frac{\tilde{Q}(\underset{\sim}{\xi},p)\exp(-i\underset{\sim}{\xi}\underset{\sim}{x})dW(\underset{\sim}{\xi})}{\underset{\sim}{\xi}.\underset{\sim}{\xi}+p/\kappa} - \frac{1}{\kappa}\int_{E_3} \frac{\tilde{Q}(\underset{\sim}{\xi}.p)\exp(-i\underset{\sim}{\xi}.\underset{\sim}{x})dW(\underset{\sim}{\xi})}{\underset{\sim}{\xi}.\underset{\sim}{\xi}+p^2\sigma^2}\} \tag{4.14''}$$

The solution of the heat-conduction equation $(4.12)_1$ has the form

$$\bar{T}(\underset{\sim}{x},p) = \frac{m}{\kappa}(2\pi)^{-3/2}\int_{E_3} \frac{\tilde{Q}\exp(-i\underset{\sim}{\xi}.\underset{\sim}{x})dW(\underset{\sim}{\xi})}{\underset{\sim}{\xi}.\underset{\sim}{\xi}+p/\kappa} \tag{4.15}$$

The first integral on the right-hand side of equation (4.14") is equal to $-\dfrac{m}{p^2\sigma^2-p/\kappa}\bar{T}$. The second integral may be regarded as the solution of the differential equation

$$(\nabla^2-p^2\sigma^2)\bar{F} = -\frac{\bar{Q}}{\kappa}, \qquad F(\underset{\sim}{x},0) = F(\underset{\sim}{x},0) = 0, \tag{4.16}$$

the structure of which is analogous to that of equation $(4.12)_2$. Thus the solution of equation (4.16) can be represented in the form |10|

$$\overline{\phi}(\underset{\sim}{x},p) = -\frac{m}{p^2\sigma^2-p/\kappa}(\overline{T}-\overline{F}). \tag{4.17}$$

It should be observed that for the determination of the function $\overline{\phi}$ it is sufficient to know the transform of the function T. The function $\overline{F}$ is deduced by replacing in the transform $\overline{T}$ the quantity p/κ by $p^2\sigma^2$. Putting $\sigma = 0$ in equation (4.17) we obtain the function $\overline{\phi}$ for the quasi-static case.

Suppose that an instantaneous concentrated heat source is acting at the origin.

We have to solve the equation

$$(\nabla^2 - \frac{1}{\kappa}\partial_t)T = -\frac{1}{\kappa}\delta(\underset{\sim}{x})\delta(t). \tag{4.18}$$

Performing the Laplace transform on equation (4.18) and assuming that $T(\underset{\sim}{x},0) = 0$, we obtain

$$(\nabla^2 - \frac{p}{\kappa})\overline{T} = -\frac{1}{\kappa}\delta(\underset{\sim}{x}). \tag{4.19}$$

The solution of this equation is represented by means of the Fourier-Hankel transform (in the cylindrical coordinate system (r,z))

$$\overline{T} = \frac{1}{2\pi^2\kappa}\int_0^\infty\int_0^\infty \frac{\alpha J_0(\alpha r)\cos\zeta z}{\alpha^2+\zeta^2+p/\kappa}d\alpha d\zeta \tag{4.20}$$

$$= \frac{1}{4\pi\kappa}\int_0^\infty \frac{\exp(-z\sqrt{\alpha^2+p/\kappa})}{\sqrt{\alpha^2+p/\kappa}}\alpha J_0(\alpha r)d\alpha$$

$$= \frac{1}{4\pi\kappa R}\exp(-R\sqrt{\frac{p}{\kappa}}).$$

Representing the function ϕ by the Fourier-Hankel integral

$$\phi = \int_0^\infty \int_0^\infty C(\alpha,\zeta,p)J_0(\alpha r)\cos\zeta z\, d\alpha\, d\zeta, \tag{4.21}$$

and inserting the last result together with $(4.20)_2$ into equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} - \sigma^2 p^2\right)\overline{\phi} = m\overline{T} \tag{4.22}$$

we find that

$$C(\alpha,\zeta,p) = -\frac{m}{2\pi^2\kappa}\frac{\alpha}{(\alpha^2+\zeta^2+p/\kappa)(\alpha^2+\zeta^2+p^2\sigma^2)}. \tag{4.23}$$

Substituting (4.23) into (4.21) we obtain after integration

$$\overline{\phi} = \frac{m}{4\pi\kappa\sigma^2 p(p-{}^1/\sigma^2\kappa)R}\left(\exp(-Rp\sigma)-\exp(-R\sqrt{\tfrac{p}{\kappa}})\right). \tag{4.24}$$

The latter result can also be derived by making use of formulae (4.17). The function $\overline{F}$ is obtained by replacing in the expression for $\overline{T}$ $(4.20)_3$ the quantity $\sqrt{\frac{p}{\kappa}}$ by $p\sigma$.

Inverting the Laplace transform in (4.24) we have [74]

$$\phi = \frac{K}{R}\left[E_1(R,t) + E_2(R,t)H(t-R\sigma)\right],$$

where

$$E_1(\zeta,\tau) = \operatorname{erfc}\left(\frac{\zeta}{2\sqrt{\tau}}\right) - \tfrac{1}{2}e^{\tau}\left(e^{\zeta}\operatorname{erfc}\left(\frac{\zeta}{2\sqrt{\tau}} + \sqrt{\tau}\right)+e^{-\zeta}\operatorname{erfc}\left(\frac{\zeta}{2\sqrt{\tau}} - \sqrt{\tau}\right)\right)$$

$$E_2(\zeta,\tau) = \exp(\tau-\zeta) - 1,$$

$$\zeta = \frac{R}{\kappa\sigma}, \qquad \tau = \frac{t}{\kappa\sigma 2}. \tag{4.25}$$

The knowledge of the functions ϕ makes it possible for us to calculate the components of the stress tensor σ_{ij} through the equations

$$\sigma_{RR} = -\frac{4\mu}{R}\frac{\partial\phi}{\partial R} + \rho\ddot{\phi},$$

$$\sigma_{\phi\phi} = \sigma_{\theta\theta} = -2\mu\left(\frac{1}{R}\frac{\partial\phi}{\partial R} + \frac{\partial^2\phi}{\partial R^2}\right) + \rho\ddot{\phi},$$

$$\sigma_{R\phi} = \sigma_{R\theta} = \sigma_{\phi\theta} = 0. \tag{4.26}$$

Introducing the notations

$$\sigma_{RR} = \sigma'_{RR}H(\sigma R-t) + \sigma''_{RR}H(t-\sigma R),$$

$$\sigma_{\phi\phi} = \sigma'_{\phi\phi}H(\sigma R-t) + \sigma''_{\phi\phi}H(t-\sigma R), \tag{4.27}$$

and

$$\rho = R/_{\kappa\sigma}, \qquad \tau = {}^{t}/\kappa\sigma^2, \quad \eta = {}^{\rho}/4\mu\sigma^2,$$

we arrive at the following formulae for the functions σ_{RR}, $\sigma_{\phi\phi}$, $\sigma_{\theta\theta}$.

$$\sigma'_{RR} = \frac{4\mu K}{\kappa^3\sigma^3\zeta^3}\left\{\operatorname{erfc}\left(\frac{\zeta}{2\sqrt{\tau}}\right) + \tfrac{1}{2}e^{\tau}\left[(1-\zeta^3+\eta\zeta^2)e^{\zeta}\operatorname{erfc}\left(\frac{\zeta}{2\sqrt{\tau}} + \sqrt{\tau}\right) + \right.\right.$$

$$\left.\left. + (1+\zeta+\eta\zeta^2)e^{-\zeta}\operatorname{erfc}\left(\frac{\zeta}{2\sqrt{\tau}} - \sqrt{\tau}\right)\right] - \frac{\eta\zeta^3}{2\sqrt{\pi\tau^3}}\exp\left(-\frac{\zeta^2}{4\tau}\right)\right\}$$

$$\sigma''_{RR} = \sigma'_{RR}(\zeta,\tau) + \frac{4\mu K}{\kappa^3\sigma^3\zeta^3}\left[(1+\zeta-\eta\zeta^2)\exp(\tau-\zeta)-1\right],$$

$$\sigma'_{\phi\phi}(\zeta,\tau) = -\frac{2\mu K}{\kappa^3\sigma^3\zeta^3}\left\{\operatorname{erfc}\left(\frac{\zeta}{2\sqrt{\tau}}\right) - \tfrac{1}{2}e^{\tau}\left[\left[1-\zeta+\zeta^2(1-2\eta)\right]e^{\zeta}\operatorname{erfc}\left(\frac{\zeta}{2\sqrt{\tau}} + \sqrt{\tau}\right) + \right.\right.$$

$$\left. + \left[1+\zeta+\zeta^2(1-2\eta)\right]e^{-\zeta}\operatorname{erfc}\left(\frac{\zeta}{2\sqrt{\tau}} - \sqrt{\tau}\right)\right] +$$

$$\left. + \frac{\eta\zeta^3}{\sqrt{\pi\tau^3}}\exp\left(-\frac{\zeta^2}{4\tau}\right)\right\}, \tag{4.28}$$

$$\sigma''_{\phi\phi}(\rho,\tau) = \sigma'_{\phi\phi}(\zeta,\tau) - \frac{2\mu K}{\kappa^3\sigma^3\zeta^3}\left(\left[1+\zeta+\zeta^2(1-2\eta)\right]\exp(\tau-\zeta)-1\right).$$

The potential ϕ is a continuous function, while the displacement $u_R = \frac{\partial\phi}{\partial R}$ suffers a discontinuity of amount $-\frac{K}{\kappa\sigma R}$ at the instant $t = R\sigma$.

At $t = R\sigma$ the stresses σ_{RR} and $\sigma_{\phi\phi}$ suffer the discontinuities $\frac{4\mu K}{\kappa^3\sigma^3}\frac{1-\eta\zeta}{\zeta^2}$ and $\frac{-2\mu K}{\kappa^3\sigma^3}\frac{1+(1-2\eta)\zeta}{\zeta^2}$ respectively. It is evident that in contrast to plane heat sources the jumps in the stresses decrease as ζ increases. We also observe that for $t \gg R\sigma$ the stresses rapidly approach their quasistatic values. In fact, for $t \gg R\sigma$ we have

$$E_1 \sim \operatorname{erfc}\left(\frac{\zeta}{2\sqrt{\tau}}\right) - \exp(\tau), \quad E_2 \sim \exp(\tau) - 1 \tag{4.29}$$

hence

$$\phi \approx -\frac{K}{R}\operatorname{erf}\left(\frac{R}{\sqrt{\vartheta}}\right) = -\frac{K}{\kappa\sigma\zeta}\operatorname{erf}\left(\frac{\zeta}{2\sqrt{\tau}}\right). \tag{4.30}$$

We now consider the plane problem of the action of a continuous line heat source in an infinite space. The starting point of our considerations are the equations:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{\kappa}\frac{\partial}{\partial t}\right)T = -\frac{\delta(r)}{2\pi r}H(t), \quad \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}\right)\phi = mT. \tag{4.31}$$

In this case it is convenient to apply to equations (4.31) successively the Laplace and Hankel transforms. Thus we obtain

$$\overline{T} = \frac{1}{2\pi\kappa p}\int_0^\infty \frac{\alpha J_0(\alpha r)d\alpha}{\alpha^2+p/\kappa} = \frac{1}{2\pi\kappa p}K_0\left(r\sqrt{\frac{p}{\kappa}}\right), \tag{4.32}$$

and

$$\begin{aligned}\overline{\phi} &= -\frac{m}{2\pi\kappa p}\int_0^\infty \frac{\alpha J_0(\alpha r)d\alpha}{(\alpha^2+p/\kappa)(\alpha^2+p^2\sigma^2)} \\ &= -\frac{mc_1^2}{2\pi\kappa p^2(p-c_1^2/\kappa)}\left(K_0\left(r\sqrt{\frac{p}{\kappa}}\right) - K_0\left(\frac{rp}{c_1}\right)\right).\end{aligned} \tag{4.33}$$

The Laplace transform of the displacement and the temperature have the form

$$\overline{u}_r(rp) = \frac{\partial\overline{\phi}}{\partial r} = -\frac{mc_1}{2\pi\kappa p(p-c_1^2/\kappa)}\left(K_1\left(\frac{rp}{c_1}\right) - \frac{c_1}{\sqrt{p\kappa}}K_1\left(r\sqrt{\frac{p}{\kappa}}\right)\right). \tag{4.34}$$

Taking into account that

$$\mathcal{L}^{-1}\left[K_0\left(r\sqrt{\frac{p}{\kappa}}\right)\right] = \frac{1}{2t}\exp\left(-\frac{r^2}{4\kappa t}\right)$$

we obtain

$$T(r,t) = \frac{1}{4\pi\kappa}\int_0^t \exp\left(-\frac{r^2}{4\kappa\tau}\right)\frac{d\tau}{\tau} = -\frac{1}{4\pi\kappa}E\left(-\frac{r^2}{4\kappa t}\right) \qquad (4.35)$$

where $Ei(-z)$ is the integral

$$Ei(-z) = \int_{-\infty}^{z} \frac{e^{-v}}{v}dv.$$

We confine ourselves to the determination of the displacement u_r for small values of time t; we may therefore replace $\overline{u}_r$ in equation (4.34) for large value of the parameter p by the asymptotic expression

$$\overline{u}_r(r,p) \sim -\frac{mc_1}{2\pi\kappa}\left(\frac{1}{p^2}K_1\left(\frac{rp}{c_1}\right) - \frac{c_1}{\sqrt{\kappa p^5}}K_1\left(r\sqrt{\frac{p}{\kappa}}\right)\right). \qquad (4.36)$$

Inverting the Laplace transform we obtain [87]

$$u_r(r,t) = \frac{m}{2\pi\kappa}\left[U_1(r,t) + U_2(r,t)H(t - {}^{r}/c_1)\right] \qquad (4.37)$$

where the functions $U_1(r,t)$ and $U_2(r,t)$ are given by the formulae

$$U_1(r,t) = \frac{c_1^2 rt}{8\kappa}\left((1 + \frac{4\kappa t}{r^2})\exp(\frac{-r^2}{4\kappa t}) + (2 + \frac{r^2}{4\kappa t})Ei(\frac{-r^2}{4\kappa t})\right),$$

$$U_2(r,t) = -\tfrac{1}{2}\,\frac{c_1 t}{r}\{\sqrt{c_1^2 t - r^2} + r\left[\log r - \log(c_1 t + \sqrt{c_1^2 t^2 - r^2}\right]\} \qquad (4.38)$$

The first term of equation (4.37) corresponds to diffusion i.e. an instantaneous displacement in the disc, while the second represnts an elastic cylindrical wave moving with the velocity c_1.

In the case of a plane source of heat situated on the plane $x_1 = 0$, the dynamic thermoelastic problem is described by the system of equations [20]

$$(\partial_1^2 - \frac{1}{\kappa}\partial_t)T = -\frac{1}{\kappa}\delta(x_1)\delta(t), \quad (\partial_1^2 - \frac{1}{c_1^2}\partial_t^2)\phi = mT. \tag{4.39}$$

Performing the Laplace transform on equations (4.39) and assuming that $T(x_1,0) = 0$, $\phi(x_1,0) = 0$, $\dot{\phi}(x_1,0) = 0$ we obtain the system

$$(\partial_1^2 - \frac{p}{\kappa})\overline{T} = -\frac{1}{\kappa}\delta(x_1), \quad (\partial_1^2 - \sigma^2 p^2)\overline{\phi} = m\overline{T}, \quad \sigma^2 = 1/c_1^2. \tag{4.40}$$

The solution of (4.40) has the form

$$\overline{T} = \frac{1}{\pi\kappa}\int_0^\infty \frac{\cos\xi_1 x_1 d\xi_1}{\xi_1^2 + p/\kappa} = \frac{1}{2\sqrt{p/\kappa}}\exp(-x_1\sqrt{\frac{p}{\kappa}}) \tag{4.41}$$

and similarly, we have for the function $\overline{\phi}$

$$\overline{\phi} = -\frac{m}{\kappa\pi}\int_0^\infty \frac{\cos\xi_1 x_1 d\xi_1}{(\xi_1^2 + p/\kappa)(\xi_1^2 + \sigma^2 p^2)} \tag{4.42}$$

$$= -\frac{m}{2\kappa\sigma^2 p}\frac{1}{(p - 1/\kappa\sigma^2)}\left(\sqrt{\frac{\kappa}{p}}\exp(-x_1\sqrt{\frac{p}{\kappa}}) - \frac{1}{\sigma p}\exp(-x_1\sigma p)\right).$$

After inverting the transform

$$T(x_1,t) = \frac{1}{\sqrt{\pi\vartheta}}\exp(-\frac{x_1^2}{\vartheta}), \quad \vartheta = 4\kappa t . \tag{4.43}$$

Let us now perform the inverse transform on the function $p^2\overline{\phi}$ to obtain [77]

$$\ddot{\phi} = -\frac{m}{2\kappa\sigma^2}\left[F_1(x_1,t) - F_2(x_1,t)H(t - x_1\sigma)\right], \tag{4.44}$$

where

$$F_1(x_1,t) = \frac{1}{2}e^\tau\left(\exp(-\zeta)\operatorname{erfc}(\frac{\zeta}{2\sqrt{\tau}} - \sqrt{\tau}) - \exp(\zeta)\operatorname{erfc}(\frac{\zeta}{2\sqrt{\tau}} + \sqrt{\tau})\right)$$

$$+ \frac{1}{\sqrt{\tau}}\exp(-\frac{\zeta}{2\sqrt{\tau}});$$

$$F_2(x_1,t) = \exp\left[\tau - \zeta\right], \quad \zeta = \frac{x_1}{\kappa\sigma}, \quad \tau = \frac{t}{\kappa\sigma^2}.$$

The stresses are calculated by means of the formulae

$$\sigma_{11} = \rho\ddot{\phi}, \quad \sigma_{22} = \sigma_{33} = -2\mu\phi_{,11} + \rho\ddot{\phi} = -2\mu mT + \frac{\lambda}{c_1^2}\ddot{\phi}, \qquad (4.45)$$

$$\sigma_{12} = \sigma_{13} = \sigma_{23} = 0.$$

It is evident that to determine the stresses it is sufficient to know the functions T and $\ddot{\phi}$.

The function $\ddot{\phi}$ and hence the stresses are represented by different formulae for $t < x_1\sigma$ and for $t > x_1\sigma$. Consider now in detail the stress $\sigma_{11} = \rho\ddot{\phi}$. The function $F_1(x_1,t)$ is of a diffusive nature - the corresponding part of the stress arises instantaneously in the entire elastic space. The function $F_2(x_1,t)H(t-x_1\sigma)$ indicates that an elastic wave is propagated. Let us take an arbitrary section x_1 = constant in the space. For the time interval $t < x_1\sigma$ we have at the point x_1 the stress

$$\sigma_{11} = -\frac{m\rho}{2\pi\sigma^3}F_1(x_1,t). \qquad (4.46)$$

At the instant $t = x_1\sigma$ the front of the elastic wave begins to pass, and the stress σ_{11} takes the value

$$\sigma_{11} = -\frac{m\rho}{2\kappa\sigma^3}\left[F_1(x_1,t) - F_2(x_1,t)\right], \ t > x_1\sigma. \qquad (4.47)$$

The stress σ_{11} undergoes, at the instant $t^0 = x_1\sigma$, a jump of finite value

$$\sigma_{11}(x_1,t^0+) - \sigma_{11}(x_1,t^0-) = \frac{m\rho}{2\kappa\sigma^3}.$$

We notice that this jump of stress is independent of the distance x_1. After the elastic wave is passed, the stress in the section x_1 = constant decreases rapidly and tends to its quasi-static value, i.e. as $t \to \infty$ the stress tends to zero.

5. Heat sources in an infinite space. Coupled problem.

Suppose that in the infinite space the temperature field is due to the action of non-stationary heat sources. We assume that the body forces vanish ($\underset{\sim}{X} = 0$) and that the temperature and all stress and displacement components vanish as $|x_1^2+x_2^2+x_3^2|^{\frac{1}{2}} \to \infty$ or $t \to \infty$. The displacements, stresses and temperature may be found simply by solving the wave equation (2.40)

$$\left[(\nabla^2 - \frac{1}{\kappa}\partial_t)(\nabla^2 - \frac{1}{c_1^2}\partial_t) - \frac{\varepsilon}{\kappa}\partial_t\nabla^2\right]\phi = -\frac{mQ}{\kappa}, \quad \varepsilon = \eta m\kappa. \tag{5.1}$$

The knowledge of its particular integral ϕ enables us to calculate the temperature from the formula

$$T = \frac{1}{m}(\nabla^2 - \frac{1}{c_1^2}\partial_t^2)\phi \tag{5.2}$$

and the displacements and stresses in accordance with the relations

$$u_i = \phi_{,i}, \quad \sigma_{ij} = 2\mu(\phi_{,ij} - \delta_{ij}\phi_{,kk}) + \rho\delta_{ij}\ddot{\phi} . \tag{5.3}$$

If we apply a four-dimensional Fourier transform to equation (5.1), we obtain

$$\bar{\phi}(\underset{\sim}{\xi},\tau) = -\frac{m}{\kappa}\frac{Q(\underset{\sim}{\xi},\tau)}{D(\underset{\sim}{\xi},\tau)} \tag{5.4}$$

where

$$\bar{\phi}(\underset{\sim}{\xi},\tau) = \frac{1}{4\pi^2}\int_{E_4}\phi(\underset{\sim}{x},t)\exp\left[i(\underset{\sim}{x}.\underset{\sim}{\xi}+t\tau)\right]dx_1dx_2dx_3dt \tag{5.5}$$

and

$$D(\underset{\sim}{\xi},\tau) = (\underset{\sim}{\xi}^2 - \frac{i\tau}{\kappa})(\underset{\sim}{\xi}^2 - \frac{\tau^2}{c_1^2}) - \frac{i\tau\varepsilon}{\kappa}\underset{\sim}{\xi}^2, \quad \underset{\sim}{\xi}^2 = \xi_1^2+\xi_2^2+\xi_3^2.$$

Inverting the Fourier transform for the function (5.4) we have

$$\phi(\underset{\sim}{x},t) = -\frac{m}{4\pi^2\kappa}\int_{E_4}\frac{\tilde{Q}(\underset{\sim}{\xi},\tau)}{D(\underset{\sim}{\xi},\tau)}\exp\left[-i(\underset{\sim}{x}\underset{\sim}{\xi}+t\tau)\right]d\xi_1 d\xi_2 d\xi_3 d\tau. \tag{5.6}$$

If we are dealing with an instantaneous concentrated heat source $Q(\underset{\sim}{x},t) = \delta(\underset{\sim}{x})\delta(t)$ then

$$Q(\underset{\sim}{\xi},\tau) = \frac{1}{4\pi^2}\int_{E_4}\delta(\underset{\sim}{x})\delta(t)\exp\left[i(\underset{\sim}{x}.\underset{\sim}{\xi}+t\tau)\right]dx_1 dx_2 dx_3 dt = \frac{1}{4\pi^2} \tag{5.7}$$

If the concentrated heat source varies harmonically with time $Q(\underset{\sim}{x},t) = \delta(\underset{\sim}{x})e^{i\omega t}$, then

$$\tilde{Q}(\underset{\sim}{\xi},\tau) = \frac{1}{4\pi^2}\int_{E_4}\delta(\underset{\sim}{x})\exp\left[i(\underset{\sim}{x},\underset{\sim}{\xi})+i(t\tau+\omega t)\right]dx_1 dx_2 dx_3 dt \tag{5.8}$$

$$= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\exp\left[i(\tau t+\omega t)\right]dt = \frac{1}{2\pi}\delta(\tau+\omega),$$

since

$$\int_{-\infty}^{\infty} e^{i\eta t}dt = 2\pi\delta(\eta).$$

Finally, if the heat source is moving in the direction x_3 with the constant velocity v, then

$$\tilde{Q}(\underset{\sim}{\xi},\tau) = \frac{1}{4\pi^2}\iint_{-\infty}^{\infty}\delta(x_1)\delta(x_2)\exp\left[i(\xi_1 x_1+\xi_2 x_2)\right]dx_1 dx_2 \times$$

$$\times\iint_{-\infty}^{\infty}\delta(x_3-vt)\exp\left[i(\xi_3 x_3+t\tau)\right]dx_3 dt$$

$$= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}e^{it}(\tau+\xi_3 v)dt = \frac{1}{2\pi}\delta(\tau+\xi_3 v). \tag{5.9}$$

In the case of an axisymmetric distribution of temperature, the field equation (5.1) should be solved in cylindrical coordinates. If we apply the Hankel-Fourier transform to (5.1), we obtain

$$\phi(\alpha,\zeta,\tau) = -\frac{m}{\kappa}\frac{Q(\alpha,\zeta,\tau)}{D(\alpha,\zeta,\tau)} \tag{5.10}$$

where

$$D(\alpha,\zeta,\tau) = \left(\alpha^2+\zeta^2-\frac{i\tau}{\kappa}\right)\left(\alpha^2+\zeta^2-\frac{\tau^2}{c_1^2}\right) - \frac{i\tau\varepsilon}{\kappa}(\alpha^2+\zeta^2).$$

Here

$$\tilde{\phi}(\alpha,\zeta,\tau) = \frac{1}{2\pi}\iint_{-\infty}^{\infty}\exp\left[i(\zeta z+t\tau)\right]dz\,dt\int_0^{\infty}\phi(r,z,t)rJ_0(\alpha r)dr. \tag{5.11}$$

Inverting the transform (5.10), we have

$$\phi(r,z,t) = -\frac{m}{2\pi\kappa}\iint_{-\infty}^{\infty}\exp\left[-i(\zeta z+t\tau)\right]d\zeta d\tau\int_0^{\infty}\frac{\tilde{Q}(\alpha,\zeta,\tau)}{D(\alpha,\zeta,\tau)}\alpha J_0(\alpha r)d\alpha, \tag{5.12}$$

and knowing the function ϕ we calculate the displacements from the formulae

$$u_r = \frac{\partial\phi}{\partial r},\quad u_z = \frac{\partial\phi}{\partial z}. \tag{5.13}$$

In the case of an instantaneous source of heat $Q(r,z,t) = \delta(z)\frac{\delta(r)}{2\pi r}\delta(t)$ we obtain

$$\tilde{Q}(\alpha,\zeta,\tau) = \frac{1}{2\pi}\iint_{-\infty}^{\infty}\delta(z)\delta(t)\exp\left[i(\zeta z+t\tau)\right]dz\,d\tau\int_0^{\infty}\frac{\delta(r)}{2\pi r}rJ_0(\alpha r)dr = \tag{5.14}$$

$$= \frac{1}{4\pi^2}.$$

If $Q(r,z,t) = \frac{\delta(r)}{2\pi r}\delta(z)e^{i\omega t}$, then

$$\tilde{Q}(\alpha,\zeta,\tau) = \frac{1}{2\pi}\delta(\tau+\omega). \tag{5.15}$$

If the concentrated heat source is now moving with a constant velocity in the direction z, then

$$Q(r,z,t) = \frac{\delta(r)}{2\pi r}\delta(z-vt).$$

We obtain for these cases

$$\tilde{Q}(\alpha,\zeta,\tau) = \frac{1}{2\pi}\delta(\tau+\zeta v). \tag{5.16}$$

For a concentrated source the intensity of which varies harmonically with time the function ϕ is obtained in the closed form

$$\phi = -\frac{me^{i\omega t}}{2\pi^2\kappa}\int_0^\infty\int_0^\infty \frac{\alpha J_0(\alpha r)\cos\zeta z}{D(\alpha,\zeta)}d\alpha d\zeta, \tag{5.17}$$

where

$$D(\alpha,\zeta) = (\alpha^2+\zeta^2) + (\alpha^2+\zeta^2)\left[q(1+\varepsilon)-\sigma^2\right] - q\sigma^2$$
$$= (\alpha^2+\zeta^2+k_1^2)(\alpha^2+\zeta^2+k_2^2)$$

and

$$k_1^2+k_2^2 = q(1+\varepsilon)-\sigma^2, \quad k_1^2k_2^2 = -q\sigma^2, \quad \varepsilon = \eta m\kappa,$$
$$q = i\omega/_\kappa, \qquad \sigma^2 = {}^{\omega^2}/c_1^2.$$

Here k_1, k_2 are roots of the equation

$$k^4+k^2\left\{\sigma^2-q(1+\varepsilon)\right\} - \sigma^2 q = 0, \tag{5.18}$$

where

$$k_1^2k_2^2 = \frac{c_1^2}{2\kappa^2}\left[-\lambda^2+i\lambda(1+\varepsilon)\pm\Delta\right],$$
$$\Delta = \left[\lambda^2(\lambda^2-(1+\varepsilon)^2)+2i\lambda^3(1-\varepsilon)\right]^{\frac{1}{2}}. \tag{5.19}$$

Here $\lambda = {}^{\omega}/_{\omega^*}$ is dimensionless and $\omega^* = {}^{c_1^2}/\kappa$ is a quantity characteristic of the thermoelastic medium, introduced by P. Chadwick and I.N. Sneddon [16].

After carrying out the integration in (5.17) we obtain [76]

$$\phi = \frac{me^{i\omega t}}{4\pi\kappa R(k_1^2-k_2^2)}\,(e^{-k_1 R}-e^{-k_2 R}) \tag{5.20}$$

where

$$R = (r^2+z^2)^{\frac{1}{2}},\ r = (x_1^2+x_2^2)^{\frac{1}{2}},\ k_{1,2} = a_{1,2}+ ib_{1,2}\quad a_{1,2} > 0.$$

From equation (5.2) we obtain

$$T = \frac{e^{i\omega t}}{4\pi\kappa R(k_1^2-k_2^2)}\Big((\sigma^2+k_1^2)e^{-k_1 R}-(\sigma^2+k_2^2)e^{-k_2 R})\Big) \tag{5.21}$$

In the particular case of zero coupling ($\varepsilon = 0$), we have

$$T = \frac{1}{4\pi\kappa R}\exp(i\omega t-R\sqrt{q}),\quad \phi = \frac{me^{i\omega t}}{4\pi\kappa R(\sigma^2+q)}\,(e^{-R\sqrt{q}}-e^{-R_2\sigma}).$$

6. Discontinuous temperature field in a semi-space [77].

Suppose that the temperature is equal to unity in a semi-infinite circular cylinder, and that it is zero outside this cylinder. Assume also that the plane $z = 0$ bounding the elastic semi-space is free of tractions. The stresses σ_{ij} due to the action of the discontinuous temperature field can be represented as the sum of two parts, the first is due to the action of the temperature field

$$T = H(a-r)\left[H(z)-H(-z)\right] \tag{6.1}$$

in the infinite space. The second state $\bar{\bar{\sigma}}_{ij}$ is so chosen in the elastic semi-space that the boundary conditions on the plane $z = 0$ (vanishing of traction) are satisfied. The stresses $\bar{\sigma}_{ij}$ due to the action of temperature field (6.1) were examined in the preceding section 1.3; they are given by formulae (3.16) of 1.3 which should be multiplied by 2. It is evident that for $z = 0$ we have $\bar{\sigma}_{zz} = 0$, and

$$\left[\sigma_{rz}\right]_{z=0} = 2\mu m a\int_0^\infty J_1(\alpha a)J_1(\alpha r)\mathrm{d}\alpha. \tag{6.2}$$

The components of the stress tensor σ_{ij} are determined by means of the Love function ϕ, i.e. [5]

$$\bar{\bar{\sigma}}_{rr} = \frac{2\mu}{1-2\nu}\frac{\partial}{\partial z}(\nu\nabla^2 - \frac{\partial^2}{\partial r^2})\phi,\ \bar{\bar{\sigma}}_{zz} = \frac{2\mu}{1-2\nu}\frac{\partial}{\partial z}\left((2-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\phi,$$
$$\bar{\bar{\sigma}}_{\phi\phi} = \frac{2\mu}{1-2\nu}\frac{\partial}{\partial z}(\nu\nabla^2 - \frac{1}{r}\frac{\partial}{\partial r})\phi,\ \bar{\bar{\sigma}}_{rz} = \frac{2\mu}{1-2\nu}\frac{\partial}{\partial r}\left((1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\phi. \tag{6.3}$$

The function ϕ satisfies the biharmonic equation

$$\nabla^2\nabla^2\phi = 0 \tag{6.4}$$

Representing ϕ by the Hankel integral

$$\phi = \int_0^\infty (C+\alpha zD)e^{-\alpha z}J_0(\alpha r)\mathrm{d}\alpha, \tag{6.5}$$

which vanishes at infinity, we have to determine the quantities C and D from the boundary conditions

$$\bar{\sigma}_{rz}+\bar{\bar{\sigma}}_{rz} = 0, \quad \bar{\bar{\sigma}}_{zz} = 0, \text{ for } z = 0. \tag{6.6}$$

The second condition yields $C = -(1-2\nu)0$ and the first

$$D = ma(1-2\nu)\frac{J_1(\alpha a)}{\alpha^3}. \tag{6.7}$$

Inserting D and C into relations (6.3) and (6.5)

$$\bar{\bar{\sigma}}_{rr} = \frac{2\mu}{1-2\nu}\int_0^\infty D(\alpha)e^{-\alpha z}\left((2-\alpha z)J_0(\alpha r)+(2\nu-2+\alpha z)\frac{J_1(\alpha r)}{\alpha r}\right)\mathrm{d}\alpha,$$

$$\sigma_{\phi\phi} = \frac{2\mu}{1-2\nu}\int_0^\infty D(\alpha)e^{-\alpha z}\alpha^3\left(2\nu J_0(\alpha r)-(2\nu-2+\alpha z)\frac{J_1(\alpha r)}{\alpha r}\right)\mathrm{d}\alpha,$$

$$\bar{\sigma}_{zz} = \frac{2\mu}{1-2\nu} z \int_0^\infty D(\alpha)\alpha^4 e^{-\alpha z} J_0(\alpha r) d\alpha ,$$

$$\bar{\bar{\sigma}}_{rz} = -\frac{2\mu}{1-2\nu} \int_0^\infty D(\alpha)\alpha^3 e^{-\alpha z} J_1(\alpha r)(1-\alpha z) d\alpha , \tag{6.8}$$

we arrive at the components of the stress tensor $\bar{\bar{\sigma}}_{ij}$. Adding the stresses $\bar{\sigma}_{ij}$ and $\bar{\bar{\sigma}}_{ij}$ we obtain the final state of stress

$$\sigma_{rr} = 2\mu m a \left\{ \int_0^\infty e^{-\alpha z} J_1(\alpha a) \left[(1-\alpha z) J_0(\alpha r) + (2\nu - 1 + \alpha z) \frac{J_1(\alpha r)}{\alpha r} \right] d\alpha - \right.$$
$$\left. - \int_0^\infty J_1(\alpha a) \frac{J_1(\alpha r)}{\alpha r} d\alpha \right\},$$

$$\sigma_{\phi\phi} = 2\mu m a \left\{ \int_0^\infty e^{-\alpha z} J_1(\alpha a) \left(2\nu J_0(\alpha r) - (2\nu - 1 + \alpha z) \frac{J_1(\alpha r)}{\alpha r} \right) d\alpha - \right.$$
$$\left. - \int_0^\infty J_1(\alpha a) \left(J_0(\alpha r) - \frac{J_1(\alpha r)}{\alpha r} \right) d\alpha \right\},$$

$$\sigma_{zz} = 2\mu m a \left\{ \int_0^\infty e^{-\alpha z} (1+\alpha z) J_1(\alpha a) J_0(\alpha r) d\alpha - \int_0^\infty J_1(\alpha a) J_0(\alpha r) d\alpha \right\},$$

$$\sigma_{rz} = 2\mu m a z \int_0^\infty e^{-\alpha z} J_1(\alpha a) J_1(\alpha r) \alpha d\alpha . \tag{6.9}$$

Making use of the functions I_1, I_2 and I_3 of the section 3 we reduce relations (6.9) to the forms

$$\sigma_{rr} = 2\mu m a \left\{ I_3(a,r,z) + z \frac{\partial}{\partial a} I_1(a,r,z) - \frac{1-2\nu}{r} \left[a I_3(r,a,z) \right. \right.$$
$$\left. \left. + r I_3(a,r,z) - z I_2(a,r,z) \right] + \frac{z}{r} I_2(a,r) - \frac{1}{2a} \left[H(a-r) + \left(\frac{a}{r}\right)^2 H(r-a) \right] \right\},$$

$$\sigma_{\phi\phi} = 2\mu m a\{2\nu I_3(a,r,z)+ \frac{1-2\nu}{2r}\left[aI_3(r,a,z)+rI_3(a,r,z)\right] -$$

$$- zI_2(a,r,z)- \frac{z}{r}I_2(a,r,z)- \frac{1}{2a}\left[H(a-r)-(\frac{a}{r})^2 H(r-a)\right] \},$$

$$\sigma_{zz} = 2\mu m a\left(I_3(a,r,z)- z\frac{\partial}{\partial a}I_1(a,r,z)- \frac{1}{a}H(a-r)\right),$$

$$\sigma_{rz} = -2\mu m a z\frac{\partial}{\partial z}I_2(a,r,z) = -2\mu m z^2(\pi^2 a)^{-\frac{1}{2}}r^{-3/2}Q'_{\frac{1}{2}}(\frac{r^2+a^2+z^2}{2ar}),$$

where $Q'_{\frac{1}{2}}(\omega)$ is the derivative of the function $Q_{\frac{1}{2}}(\omega)$ with respect to the argument ω. For $z = 0$

$$\sigma_{rr}(r,0) = 2\mu m\left(\nu H(a-r)-(1-\nu)\frac{a^2}{r^2}H(r-a)\right),$$

$$\sigma_{\phi\phi}(r,0) = 2\mu m\left(\nu H(a-r)+(1+\nu)\frac{a^2}{r^2}H(r-a)\right), \tag{6.11}$$

$$\sigma_{rz}(r,0) = \sigma_{zz}(r,0) = 0.$$

Along the z-axis the stresses are

$$\sigma_{rr}(0,z) = \mu m a\left[(1+2\nu)\overline{\kappa}(z)+z\overline{\kappa}'(z)-\overline{\kappa}(0)\right],$$

$$\sigma_{\phi\phi}(0,z) = \sigma_{rr}(0,z),$$

$$\sigma_{zz}(0,z) = 2\mu m a\left[1-z\kappa'(z)-\overline{\kappa}(0)\right],$$

$$\sigma_{rz}(0,z) = 0, \tag{6.12}$$

where

$$\overline{\kappa}(z) = \int_0^\infty e^{-\alpha z}J_1(\alpha a)d\alpha = \frac{1}{a}\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right).$$

For $z \to \infty$, we obtain a plane state of stress; its components are independent of the variable z and $\sigma_{rz}(r,\infty) = 0$. The stresses in this case are the stresses $\sigma_{ij}^{(2)}$ represented by relations (3.17) of 3 multiplied by 2.

7. Stresses in an elastic half-space due to a heat exposure on the bounding plane [64, 62, 110, 98, 76].

We consider an elastic half-space on the boundary of which the temperature $T(x_1,x_2,0) = f(x_1,x_2)$ is prescribed, and assume that there are no heat sources inside the half-space. It can be proved that if the boundary is free of tractions then a plane state of stress parallel to the boundary $x_3 = 0$ exists in the semi-space.

We first solve an auxiliary problem. Consider the heat conduction equation in cylindrical coordinates

$$\nabla^2 T^*(r,z) = 0 \tag{7.1}$$

with the boundary condition

$$T^*(r,0) = \frac{\delta(r)}{2\pi r}, \quad T^*_\infty = 0. \tag{7.2}$$

The solution of (7.1) satisfying the required boundary condition is given by the function

$$T^*(r,z) = \frac{1}{2\pi}\int_0^\infty \alpha e^{-\alpha z} J_0(\alpha r)\,d\alpha = \frac{z}{2\pi R^3}, \quad z > 0, \tag{7.3}$$

where

$$R = (r^2+z^2)^{\frac{1}{2}}, \qquad r^2 = x_1^2 + x_2^2,$$

is the distance of a point of the elastic semi-space from the origin of coordinates. Further, we have to solve the equation

$$\nabla^2 \phi^*(r,z) = mT^* \tag{7.4}$$

with the arbitrary boundary condition

$$\phi^* = 0 \text{ for } z = 0. \tag{7.5}$$

For this boundary condition we have $\overline{\sigma}_{zz} = 0$ for $z = 0$. The solution of the equation (7.4) has the form

$$\phi^* = \frac{m}{\pi^2}\int_0^\infty\int_0^\infty \frac{\alpha J_0(\alpha r)}{(\alpha^2+\zeta^2)^2}\zeta \sin\zeta z \, d\alpha \, d\zeta$$

$$= -\frac{zm}{4\pi}\int_0^\infty e^{-\alpha z} J_0(\alpha r)\, d\alpha = -\frac{mz}{4\pi R}, \quad z > 0. \tag{7.6}$$

Knowing the function ϕ^* we can find the stresses $\bar{\sigma}^*_{ij}$ from the formulae

$$\bar{\sigma}^*_{rr} = -2\mu\left(\frac{\partial^2}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\phi^*, \quad \bar{\sigma}^*_{\phi\phi} = -2\mu\left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2}\right)\phi^*,$$

$$\bar{\sigma}^*_{zz} = -2\mu\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\phi^*, \quad \bar{\sigma}^*_{rz} = 2\mu\frac{\partial^2\phi^*}{\partial r \partial z}, \tag{7.7}$$

or

$$\bar{\sigma}^*_{rr} = -A\int_\infty^\infty \alpha e^{-\alpha z}\left((2-\alpha z)J_0(\alpha r) + \alpha z\frac{J_1(\alpha r)}{\alpha r}\right)d\alpha,$$

$$\bar{\sigma}^*_{\phi\phi} = -A\int_0^\infty \alpha e^{-\alpha z}\left(2J_0(\alpha r) - \alpha z\frac{J_1(\alpha r)}{\alpha r}\right)d\alpha,$$

$$\bar{\sigma}^*_{zz} = -Az\int_0^\infty \alpha^2 J_0(\alpha r)e^{-\alpha z}\, d\alpha, \tag{7.8}$$

$$\bar{\sigma}^*_{rz} = A\int_0^\infty \alpha(1-\alpha z)e^{-\alpha z}J_1(\alpha r)\, d\alpha, \quad A = \frac{m\mu}{2\pi}.$$

On the boundary we have the expressions

$$\bar{\sigma}^*_{zz} = 0 \text{ and } \bar{\sigma}^*_{rz} \neq 0 \text{ for } z = 0.$$

The stresses $\bar{\bar{\sigma}}^*_{ij}$ should be so chosen that all boundary conditions on the plane $z = 0$ are satisfied

$$\bar{\bar{\sigma}}^*_{zz} = 0, \qquad \bar{\bar{\sigma}}^*_{rz} + \bar{\sigma}^*_{rz} = 0 \text{ for } z = 0. \tag{7.9}$$

The state $\bar{\sigma}^*_{ij}$ is determined by means of the formulae (6.3).

The Love function ϕ, has the form (6.5).

The first equation (7.9) yields $C = -(1-2\nu)D$; the second condition $D = \frac{1-2\nu}{2\mu\alpha^2}A$.

We obtain

$$\begin{aligned}
\bar{\bar{\sigma}}^{*}_{rr} &= A\int_0^\infty \alpha e^{-\alpha z}\left[(2-\alpha z)J_0(\alpha r)+(2\nu-2+\alpha z)\frac{J_1(\alpha r)}{\alpha r}\right]d\alpha,\\
\bar{\bar{\sigma}}^{*}_{\phi\phi} &= A\int_0^\infty \alpha e^{-\alpha z}\left[2\nu J_0(\alpha r)-(2\nu-2+\alpha z)\frac{J_1(\alpha r)}{\alpha r}\right]d\alpha,\\
\bar{\bar{\sigma}}^{*}_{zz} &= Az\int_0^\infty \alpha^2 e^{-\alpha z}J_0(\alpha r)d\alpha,\\
\bar{\bar{\sigma}}^{*}_{rz} &= -A\int_0^\infty \alpha e^{-\alpha z}(1-\alpha z)J_1(\alpha r)d\alpha.
\end{aligned} \tag{7.10}$$

Adding the stresses $\bar{\sigma}^{*}_{ij}$ and $\bar{\bar{\sigma}}^{*}_{ij}$, we obtain

$$\begin{aligned}
\sigma^{*}_{rr} &= -\frac{E\alpha_t}{2\pi r}\int_0^\infty e^{-\alpha z}J_1(\alpha r)d\alpha = -\frac{\beta}{R(R+z)},\quad \beta = \frac{E\alpha_t}{2\pi},\\
\sigma^{*}_{\phi\phi} &= -\frac{E\alpha_t}{2\pi}\int_0^\infty \alpha e^{-\alpha z}\left[J_0(\alpha r)-\frac{J_1(\alpha r)}{\alpha r}\right]d\alpha = -\beta\left(\frac{z}{R^3}-\frac{1}{R(R+z)}\right),\\
\sigma^{*}_{zz} &= 0,\quad \sigma^{*}_{rz} = 0.
\end{aligned} \tag{7.11}$$

It is of considerable importance to observe that a plane state of stress occurs in this case.

Passing to a rectangular coordinate system, we make use of the formulae

$$\sigma^*_{11} = \sigma^*_{rr}\cos^2\phi+\sigma^*_{\phi\phi}\sin^2\phi, \quad \sigma^*_{22} = \sigma^*_{rr}\sin^2\phi+\sigma^*_{\phi\phi}\cos^2\phi,$$
$$\sigma^*_{12} = \tfrac{1}{2}(\sigma^*_{rr}-\sigma^*_{\phi\phi})\sin 2\phi, \quad \sin\phi = \frac{x_2}{r}, \quad \cos\phi = \frac{x_1}{r}, \tag{7.12}$$

and we move the Dirac function from the origin of coordinates to the point $(\xi_1,\xi_2,0)$; we then obtain

$$\sigma^*_{11} = -\frac{\beta}{r^2}\left\{1-\frac{x_3}{R}+(x_2-\xi_2)^2\left[\frac{x_3}{R^3}-\frac{2}{R(R+x_3)}\right]\right\},$$
$$\sigma^*_{22} = -\frac{\beta}{r^2}\left\{1-\frac{x_3}{R}+(x_1-\xi_1)^2\left[\frac{x_3}{R^3}-\frac{2}{R(R+x_3)}\right]\right\},$$
$$\sigma^*_{12} = \frac{\beta}{r^2}\left\{\frac{x_3}{R}-\frac{2}{R(R+x_3)}\right\}(x_1-\xi_1)(x_2-\xi_2), \tag{7.13}$$
$$\sigma^*_{13} = \sigma^*_{23} = \sigma^*_{33} = 0,$$

where

$$r = \left[(x_1-\xi_1)^2+(x_2-\xi_2)^2\right]^{\frac{1}{2}}, \quad R = (r^2+x_3^2)^{\frac{1}{2}}.$$

The stresses given by formulae (7.13) correspond to the temperature field T^* satisfying the harmonic equation with the boundary condition $T^*(x_1,x_2,0;\xi_1,\xi_2,0) = \delta(x_1-\xi_1)\delta(x_2-\xi_2)$. If we are given a distribution of temperature $T(x_1,x_2,0) = f(x_1,x_2)$ on the plane $x_3 = 0$ in a region Γ, the temperature in the semi-space is represented by the formulae

$$T(x_1,x_2,x_3) = \iint_\Gamma f(\xi_1,\xi_2)T^*(x_1,x_2,x_3;\xi_1,\xi_2,0)d\xi_1 d\xi_2. \tag{7.14}$$

The stresses σ_{ij} due to the temperature field T are derived by means of the Green functions σ^*_{ij}

$$\sigma^*_{ij}(x_1,x_2,x_3) = \iint_\Gamma f(\xi_1,\xi_2)\sigma^*_{ij}(x_1,x_2,x_3;\xi_1,\xi_2,0)d\xi_1 d\xi_2. \tag{7.15}$$

We now proceed to examine the particular case of an axially symmetric problem. Suppose that on the boundary of the elastic half-space the following thermal boundary conditions are given [77]:-

$$T(r,0) = V_0 = \text{const. for } 0 < r < a,$$
$$\frac{\partial T}{\partial z} = 0 \qquad \text{for } a < r < \infty \tag{7.16}$$
$$T\infty = 0.$$

Observe that the function

$$T(r,z) = \int_0^\infty C(\alpha)e^{-\alpha z}J_0(\alpha r)d\alpha, \quad z > 0, \tag{7.17}$$

satisfies the heat conduction equation in the elastic half-space. The quantity $C(\alpha)$ should be chosen such that the boundary conditions for $z = 0$ are satisfied

$$\int_0^\infty C(\alpha)J_0(\alpha r)d\alpha = V_0 \text{ for } 0 < r < a$$
$$\int_0^\infty C(\alpha)\alpha J_0(\alpha r)d\alpha = 0 \quad \text{for } a < r < \infty. \tag{7.18}$$

Solving this system of dual integral equations, we obtain

$$C(\alpha) = \frac{2V_0 a}{\pi}\frac{\sin\alpha a}{a\alpha}, \tag{7.19}$$

The last result can also be derived by making use of the Hankel integrals

$$\int_0^\infty \frac{\sin\alpha a}{\alpha}J_0(\alpha r)d\alpha = \frac{\pi}{2}\left(H(a-r) + \frac{2}{\pi}H(r-a)\sin^{-1}\left(\frac{a}{r}\right)\right) \tag{7.20}$$
$$\int_0^\infty \sin\alpha a J_0(\alpha r)d\alpha = H(a-r)(a^2-r^2)^{-\frac{1}{2}}.$$

In determining the stresses we can proceed from the differential equation for the thermoelastic displacement potential

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\phi = mT, \tag{7.21}$$

the solution on which can be written thus

$$\begin{aligned}\phi &= -\frac{4mV_0}{\pi^2}\iint_0^\infty \frac{\sin\alpha a J_0(\alpha r)}{(\alpha^2+\zeta^2)}\cos\zeta z\,d\alpha\,d\zeta \\ &= -\frac{V_0 m}{\pi}\int_0^\infty \alpha^{-3}(1+\alpha z)e^{-\alpha z}\sin\alpha a J_0(\alpha r)\,d\alpha, \quad z > 0.\end{aligned} \tag{7.22}$$

Next, we calculate the stresses from the formulae

$$\begin{aligned}\bar{\sigma}_{rr} &= -2\mu\left(\frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2}\right), \quad \bar{\sigma}_{\phi\phi} = -2\mu\left(\frac{\partial^2\phi}{\partial r^2} + \frac{\partial^2\phi}{\partial z^2}\right), \\ \bar{\sigma}_{zz} &= -2\mu\left(\frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial r^2}\right), \quad \bar{\sigma}_{rz} = 2\mu\frac{\partial^2\phi}{\partial r\partial z}.\end{aligned} \tag{7.23}$$

Observe that, in the plane $z = 0$, we have $\bar{\sigma}_{rz} = 0$. In order to annul the stress $\bar{\sigma}_{zz}(r,0)$ a state of stress $\bar{\bar{\sigma}}_{ij}$ expressed by means of the Love function should be added to the state of stress $\bar{\sigma}_{ij}$.

The function ϕ is taken in the form

$$\phi = \int_0^\infty (C+\alpha z D)e^{-\alpha z}J_0(\alpha r)\,d\alpha, \tag{7.24}$$

the constants C and D being determined by the boundary conditions

$$\bar{\bar{\sigma}}_{rz} = 0, \quad \bar{\sigma}_{zz} + \bar{\bar{\sigma}}_{zz} = 0 \text{ for } z = 0. \tag{7.25}$$

Taking into account relations (6.3) we have

$$C = 2\nu D, \qquad D = (1-2\nu)\frac{V_0 m}{\pi}\frac{\sin\alpha a}{\alpha^4}. \tag{7.26}$$

Having found the stresses $\bar{\bar{\sigma}}_{ij}$ and completing them by $\bar{\sigma}_{ij}$ we obtain the final state of stress

$$\sigma_{rr} = -\frac{4\beta V_0}{r}\int_0^\infty \alpha^{-2}e^{-\alpha z}\sin\alpha a J_1(\alpha r)d\alpha$$

$$= \frac{4\beta V_0}{r^2}\int_0^r \eta\sin^{-1}\left(\frac{2a}{\sqrt{z^2+(a+\eta)^2}+\sqrt{z^2+(a-\eta)^2}}\right)d\eta,$$

$$\sigma_{\phi\phi} = -\sigma_{rr}-4\beta V_0 \quad \sin^{-1}\left|\frac{2a}{\sqrt{z^2+(a+\eta)^2}+\sqrt{z^2+(a-\eta)^2}}\right| \quad , \tag{7.27}$$

$$\sigma_{zz} = 0, \qquad \sigma_{rz} = 0, \qquad \beta = \frac{E\alpha_t}{2\pi}.$$

We next examine the action of heat sources distributed in the region Γ of the plane bounding the elastic semi-space. Suppose that Γ is the circle of radius a and assume that the following boundary conditions are prescribed on the plane $z = 0$.

$$\left(-2k\frac{\partial T}{\partial z}\right)_{z=0} = WH(a-r), \quad W = \text{const.} \tag{7.28}$$

$$\sigma_{rz} = \sigma_{zz} = 0 \text{ for } z = 0.$$

The first condition (7.28) can be put in the form

$$\left(-2k\frac{\partial T}{\partial z}\right)_{z=0} = Wa\int_0^\infty J_1(\alpha a)J_0(\alpha r)d\alpha. \tag{7.29}$$

It is readily verified that the function

$$T(r,z) = \frac{Wa}{2k}\int_0^\infty \frac{e^{-\alpha z}}{\alpha}J_1(\alpha a)J_1(\alpha r)d\alpha, \; z > 0, \tag{7.30}$$

satisfies the heat conduction equation. Solving equation (7.21) we find the function ϕ which yields the stresses $\bar{\sigma}_{ij}$, and by means of the Love function ϕ we satisfy the boundary conditions in the plane $z = 0$. The final stresses $\sigma_{ij} = \bar{\sigma}_{ij} + \bar{\bar{\sigma}}_{ij}$ have the form

$$\begin{aligned}
\sigma_{rr} &= -\frac{W\beta_1}{} \int_0^\infty \alpha^{-2} e^{-\alpha z} J_1(\alpha a) J_1(\alpha r) d\alpha, \\
\sigma_{\phi\phi} &= -W\beta_1 \int_0^\infty \alpha^{-1} e^{-\alpha z} \left(J_0(\alpha r) - \frac{J_1(\alpha r)}{\alpha r} \right) J_1(\alpha a) d\alpha, \qquad (7.31) \\
\sigma_{rz} &= \sigma_{zz} = 0, \quad \beta_1 = \frac{E a \alpha_t}{2k}.
\end{aligned}$$

For $z = 0$ the stresses σ_{rr}, $\sigma_{\phi\phi}$ can be expressed by means of the complete elliptic integrals

$$\sigma_{rr}(r,0) = -\frac{W\beta_1}{\pi} \frac{r+a}{r} \left(E\left(\frac{2i\sqrt{ra}}{|r-a|}\right) - K\left(\frac{2i\sqrt{ra}}{|r-a|}\right) \right), \qquad (7.32)$$

$$\begin{aligned}
\sigma_{\phi\phi}(r,0) = -\frac{2W\beta_1}{\pi} \Bigg(&\left\{ E\left(\frac{r}{a}\right) - \frac{r+a}{2r} \left[E\left(\frac{2i\sqrt{ra}}{a-r}\right) - K\left(\frac{2i\sqrt{ra}}{a-r}\right) \right] \right\} H(a-r) + \\
&+ \left\{ \frac{r}{a} \left[K\left(\frac{a}{r}\right) - \left(1 - \frac{a^2}{r^2}\right) E\left(\frac{a}{r}\right) \right] - \right. \\
&\left. - \frac{r+a}{2r} \left[E\left(\frac{2i\sqrt{ra}}{r-a}\right) - K\left(\frac{2i\sqrt{ra}}{r-a}\right) \right] \right\} H(r-a) \Bigg)
\end{aligned}$$

where

$$E(\omega) = \int_0^{\pi/2} (1-\omega^2 \sin^2\chi)^{\frac{1}{2}} d\chi = \frac{\pi}{2} {}_2F_1(-\tfrac{1}{2}, \tfrac{1}{2}; 1; \omega^2),$$

$$K(\omega) = \int_0^{\pi/2} (1-\omega^2 \sin^2\chi)^{-\frac{1}{2}} d\chi = \frac{\pi}{2} {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; \omega^2).$$

For $r = 0$, we have

$$\sigma_{rr}(0,z) = \sigma_{\phi\phi}(0,z) = -\frac{W\beta_1}{2a} \left(\sqrt{z^2+a^2} - z \right).$$

All the problems examined above were solved under the assumption that the plane $x_3 = 0$ was free from tractions, the plane state of stress occurring in the elastic semi-space being a result of this assumption. It can easily be proved that the boundary conditions

$$\sigma_{13} = \sigma_{23} = u_3 = 0 \quad \text{or} \quad u_1 = u_2 = u_3 = 0$$

in the plane $x_3 = 0$ lead to a three-dimensional state of stress.

8. Heat sources in an elastic disk [75]

We now present a method for solving the problem of stresses due to the action of heat sources in thin plates. To this end we consider the pair of equations

$$\nabla_1^2 T = -\frac{Q}{\kappa}, \qquad \nabla_1^2 \phi = m_0 T, \qquad m_0 = \frac{\gamma}{2(\lambda+\mu)}, \tag{8.1}$$

and eliminate the function T to obtain

$$\nabla_1^2 \nabla_1^2 \phi = -m_0 \frac{Q}{\kappa}, \qquad \nabla_1^2 = \partial_1^2 + \partial_2^2. \tag{8.2}$$

We chose the boundary conditions for equation (8.2) in such a way that the resulting solution is as simple as possible. The first condition follows immediately; we should have $\nabla^2\phi = 0$ if we assume that $T = 0$ on the boundary. For the second boundary condition we take $\phi = 0$. This assumption is very convenient for plates with rectilinear boundaries. If on the boundary parallel to the x_1-axis we have $\phi = 0$, $\nabla_1^2\phi = 0$ it follows that $\partial_1^2\phi = \partial_2^2\phi = 0$ on this boundary. Hence according to relations

$$\bar{\sigma}_{ij} = 2\mu(\phi_{,ij} - \delta_{ij}\phi_{,kk}), \tag{8.3}$$

we have $\bar{\sigma}_{11}(x_1,0) = \bar{\sigma}_{22}(x_1,0) = 0$ and $\bar{\sigma}_{12}(x_1,0) \neq 0$. The stress $\bar{\sigma}_{12}(x_1,0)$ may be annulled by introducing a correcting solution in which the stresses $\bar{\bar{\sigma}}_{ij}$ are obtained by superposition

$$\sigma_{ij} = \bar{\sigma}_{ij} + \bar{\bar{\sigma}}_{ij} = (\partial_i\partial_j - \delta_{ij}\nabla_1^2)(2\mu\phi - F). \tag{8.4}$$

The equation (8.2) is analogous to that for the deflection of a plate

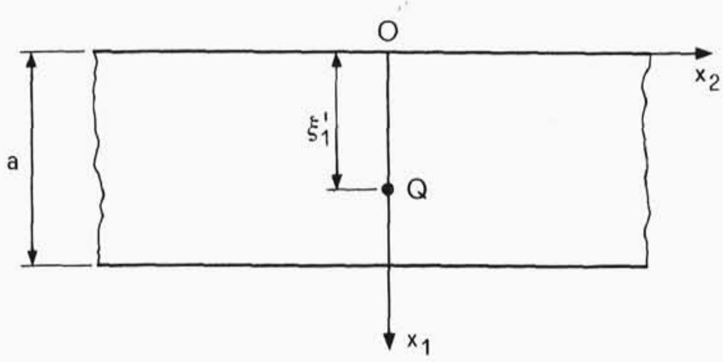

Fig. 8.1

[21, 115]

$$N\nabla_1^2\nabla_1^2 w = p. \tag{8.5}$$

For simply supported rectilinear boundaries the boundary conditions have the form $w = 0$, $\nabla^2 w = 0$.

Taking into account the analogy between equations (8.2) and (8.5) and the boundary conditions, we may use for the determination of the function ϕ the familiar solutions of plate theory.

Let us examine an example of a disc strip in which there acts a concentrated heat source of intensivity Q_0 at the point $(\xi_1',0)$, (cf. Fig. 8.1). The first stage of the analysis consists of finding the solution of the equation

$$\nabla_1^2\nabla_1^2\phi = -\frac{m_0 Q_0}{\kappa}\delta(x_1'-\xi_1')\delta(x_2). \tag{8.6}$$

With the boundary conditions

$$\begin{aligned} &\phi(0,x_2) = 0, \quad \phi(a,x_2) = 0, \\ &\nabla^2\phi(0,x_2) = 0, \quad \nabla^2\phi(a,x_2) = 0. \end{aligned} \tag{8.7}$$

It is known that the deflection of a simply supported plate strip loaded by a concentrated force P at the point $(\xi_1',0)$ is the following

$$w = \frac{2P}{a\pi N}\sum_{n=1}^{\infty}\sin\alpha_n\xi_1'\sin\alpha_n x_1' \int_0^{\infty}\frac{\cos\zeta x_2}{(\alpha_n^2+\zeta^2)^2}d\zeta, \quad \alpha_n = \frac{n\pi}{a}. \tag{8.8}$$

We can obtain the result (8.8) applying to equation (8.5) finite sine and cosine transforms. It follows from the analogy between (8.2) and (8.8) and the corresponding boundary conditions that

$$\phi = -\frac{2K}{a\pi}\sum_{n=1}^{\infty}\sin\alpha_n\xi_1'\sin\alpha_n x_1'\int_0^{\infty}\frac{\cos\zeta x_2}{(\alpha_n^2+\zeta^2)^2}d\zeta, \quad K = \frac{m_0 Q_0}{\kappa}, \tag{8.9}$$

i.e.

$$\begin{aligned} \phi &= -\frac{Ka^2}{2\pi^3}\sum_{n=1}^{\infty}\frac{e^{-\alpha_n x_2}}{n^3}(1+\alpha_n x_2)\sin\alpha_n\xi_1'\sin\alpha_n x_1' \quad \text{for } x_2 > 0, \\ \phi &= -\frac{Ka^2}{2\pi^3}\sum_{n=1}^{\infty}\frac{e^{\alpha_n x_2}}{n^3}(1-\alpha_n x_2)\sin\alpha_n\xi_1'\sin\alpha_n x_1' \quad \text{for } x_2 < 0. \end{aligned} \tag{8.10}$$

The stresses have the form

$$\begin{aligned} \bar\sigma_{1'1'} &= -2\mu\frac{\partial^2\phi}{\partial x_2^2} = -\frac{K\mu}{\pi}\sum_{n=1}^{\infty}\frac{e^{-\alpha_n x_2}}{n}(1-\alpha_n x_2)\sin\alpha_n\xi_1'\sin\alpha_n x_1', \\ \bar\sigma_{22} &= -2\mu\frac{\partial^2\phi}{\partial x_1'^2} = -\frac{K\mu}{\pi}\sum_{n=1}^{\infty}\frac{e^{-\alpha_n x_2}}{n}(1+\alpha_n x_2)\sin\alpha_n\xi_1'\sin\alpha_n x_1', \\ \bar\sigma_{1'2} &= 2\mu\frac{\partial^2\phi}{\partial x_1'\partial x_2} = \frac{K\mu}{a}x_2\sum_{n=1}^{\infty}e^{-\alpha_n x_2}\sin\alpha_n\xi_1'\cos\alpha_n x_1'. \end{aligned} \tag{8.11}$$

These formulae are valid for $x_2 > 0$. Since the summations entering formulae (8.11) are slowly convergent it is more convenient to express them in a closed form by means of the function

$$\vartheta(x_1', x_2; \xi_1', 0) = \frac{1}{4\pi}\log\frac{\cosh\frac{\pi x_2}{a} - \cos\frac{\pi}{a}(x_1'-\xi_1')}{\cosh\frac{\pi x_2}{a} - \cos\frac{\pi}{a}(x_1'+\xi_1')} \tag{8.12}$$

Namely, we have

$$\bar{\sigma}_{1'1'} = K\mu\left(\vartheta + x_2\frac{\partial}{\partial x_2}\right), \quad \bar{\sigma}_{22} = K\mu\left(\vartheta - x_2\frac{\partial}{\partial x_2}\right),$$
$$\bar{\sigma}_{1'2} = -K\mu x_2\frac{\partial}{\partial x_1'}. \tag{8.13}$$

It is easily seen that the stresses $\bar{\sigma}_{1'1'}$ and $\bar{\sigma}_{22}$ vanish on the boundary of the disc. In the vicinity of the heat source the normal stresses tend to infinity logarithmically. The normal stresses are symmetric and the stress $\bar{\sigma}_{1'2}$ is antisymmetric with respect to the x_1'-axis. On the axis $x_2 = 0$, $\bar{\sigma}_{1'2} = 0$. As $x_2 \to \infty$ all the components of stress vanish. On the boundaries $x_1' = 0$ and $x_1' = a$ the stress $\bar{\sigma}_{1'2}$ does not vanish. To determine the state of stress $\bar{\bar{\sigma}}_{ij}$ by means of the Airy function it is expedient to employ the stress not in the form (8.13) but in the form following directly from the function ϕ given by (8.11)

$$\bar{\sigma}_{1'2} = \frac{4\mu K}{a\pi}\int_0^\infty \zeta\sin\zeta x_2 \sum_{n=1}^\infty \frac{\alpha_n\sin\alpha_n\xi_1'\cos\alpha_n x_1'}{(\alpha_n^2+\zeta^2)^2}d\zeta. \tag{8.14}$$

Taking into account that

$$\sum_{n=1}^\infty \frac{\alpha_n\sin\alpha_n\xi_1'}{(\alpha_n^2+\zeta^2)} = \frac{a^3}{4}\eta_1(\xi_1',\zeta),$$
$$\sum_{n=1}^\infty \frac{(-1)^n\alpha_n\sin\alpha_n\xi_1'}{(\alpha_n^2+\zeta^2)^2} = \frac{a^3}{4}\eta_2(\xi_1',\zeta), \tag{8.15}$$

where

$$\eta_1(\xi_1',\zeta) = \frac{\zeta\xi_1'\sinh\lambda\cosh\zeta(a-\xi_1')-\lambda\sinh\zeta\xi_1'}{\lambda^2\sinh^2\lambda}$$

$$\eta_2(\xi_1',\zeta) = \frac{\zeta\xi_1'\sinh\lambda\cosh\zeta\xi_1'-\lambda\cosh\lambda\sinh\zeta\xi_1'}{\lambda^2\sinh^2\lambda}, \quad \lambda = \zeta a,$$

we obtain

$$\begin{aligned} \left[\bar{\sigma}_{1'2}\right]_{x_1=0} &= \frac{\mu K a^2}{\pi}\int_0^\infty \zeta\eta_1(\xi_1',\zeta)\sin\zeta x_2 d\zeta, \\ \left[\bar{\sigma}_{1'2}\right]_{x_1'=a} &= \frac{\mu K a^2}{\pi}\int_0^\infty \zeta\eta_2(\xi_1',\zeta)\sin\zeta x_2 d\zeta. \end{aligned} \tag{8.16}$$

The heat source acting at the point $(\xi_1',0)$ is now replaced by the action of a pair of heat sources of half intensity. We situate them first symmetrically and then antisymmetrically with respect to the x_2-axis (cf. Fig. 8.2).

For the symmetrically situated heat sources

$$\left[\bar{\sigma}_{12}^{(s)}\right]_{x_1=\frac{a}{2}} = \frac{\mu K a^2}{8\pi}\int_0^\infty \zeta\rho^{(s)}(\zeta,\xi_1)\sin\zeta x_2 d\zeta \tag{8.17}$$

where

$$\rho^{(s)}(\zeta,\xi_1) = \frac{\zeta\xi_1\cosh\mu_0\sinh\zeta\xi_1-\mu_0\sinh\mu_0\cosh\zeta\xi_1}{\mu_0^2\cosh^2\mu_0}, \quad \mu_0 = \zeta\frac{a}{2},$$

while for the antisymmetric arrangement

$$\left[\bar{\sigma}_{12}^{(a)}\right]_{x_1=\frac{a}{2}} = \frac{\mu K a^2}{8\pi}\int_0^\infty \zeta\rho^{(a)}(\zeta,\xi_1)\sin\zeta x_2 d\zeta, \tag{8.18}$$

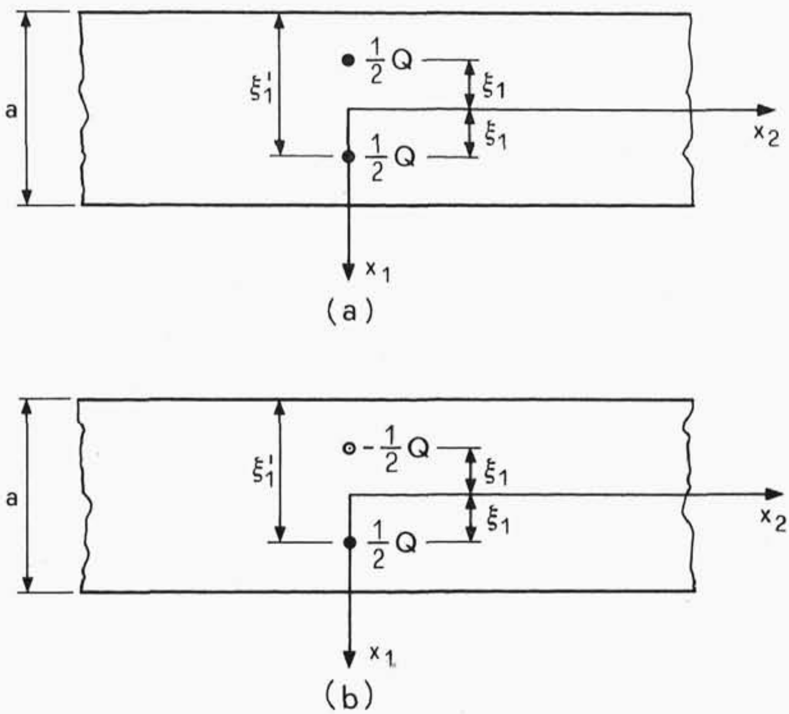

Fig. 8.2

where

$$\rho^{(a)}(\zeta,\xi_1) = \frac{\zeta\xi_1 \sinh\mu_0 \cosh\zeta\xi_1 - \mu_0 \cosh\mu_0 \sinh\zeta\xi_1}{\mu_0{}^2 \sinh^2\mu_0}$$

To annul the stresses $\overline{\sigma}_{12}^{(s)}$ occurring on the boundaries $x_1 = \pm\frac{a}{2}$ the state of stress $\overline{\sigma}_{ij}^{(s)}$ must be completed by a state $\overline{\overline{\sigma}}_{ij}^{(s)}$ expressed by means of the Airy function, which satisfies the equation

$$\nabla_1^2 \nabla_1^2 F^{(s)} = 0 \qquad (8.19)$$

and the boundary conditions

$$F^{(s)}_{,22} = 0, \quad -F^{(s)}_{,12} + \overline{\sigma}^{(s)}_{12} = 0 \text{ for } x_1 = a/2. \tag{8.20}$$

Accordingly, we take the function $F^{(s)}$ in the form

$$F^{(s)} = \int_0^\infty \zeta^{-2}(A\cosh\zeta x_1 + B\zeta x_1\sinh\zeta x_1)\cos\zeta x_2 d\zeta. \tag{8.21}$$

Having determined the constants A,B from the boundary conditions (8.20) we have

$$\begin{aligned}
\overline{\overline{\sigma}}^{(s)}_{11} &= \frac{\mu K a}{2\pi}\int_0^\infty D(\zeta,\xi_1)(\mu_0\sinh\mu_0\cosh\zeta x_1 - \zeta x_1\cosh\mu_0\cosh\zeta x_1)\cos\zeta x_2 d\zeta,\\
\overline{\overline{\sigma}}^{(s)}_{22} &= -\frac{\mu K a}{2\pi}\int_0^\infty D(\zeta,\xi_1)\big[(\mu_0\sinh\mu_0 - 2\cosh\mu_0)\cosh\zeta x_1 -\\
&\quad - \zeta x_1\cosh\mu_0\sinh\zeta x_1\big]\cos\zeta x_2 d\zeta,\\
\overline{\overline{\sigma}}^{(s)}_{12} &= -\frac{\mu K a}{2\pi}\int_0^\infty D(\zeta,\xi_1)\big[(\mu_0\sinh\mu_0 - \cosh\mu_0)\sinh\zeta x_1 -\\
&\quad - \zeta x_1\cosh\mu_0\cosh\zeta x_1\big]\sin\zeta x_2 d\zeta,
\end{aligned} \tag{8.22}$$

where

$$D(\zeta,\xi_1) = -\frac{\mu_0\rho^{(s)}(\zeta,\xi_1)}{\sinh 2\mu_0 + 2\mu_0}$$

For the sources situated antisymmetrically with respect to the x_2-axis the Airy function is taken in the form

$$F^{(a)} = \int_0^\infty \zeta^{-2}(C\sinh\zeta x_1 + D\zeta x_1\cosh\zeta x_1)\cos\zeta x_2 d\zeta, \tag{8.23}$$

and we determine quantities C, D from the boundary conditions

$$F^{(a)}_{,22} = 0, \quad -F^{(a)}_{,12} + \bar{\sigma}^{(a)}_{12} = 0 \text{ for } x_1 = a/2 \tag{8.24}$$

The stresses $\bar{\bar{\sigma}}^{(a)}_{ij}$ are

$$\bar{\bar{\sigma}}^{(a)}_{11} = \frac{\mu K a}{2\pi}\int_0^\infty G(\zeta,\xi_1)(\mu_0\cosh\mu_0\sinh\zeta x_1 - \tag{8.25}$$

$$- \zeta x_1 \sinh\mu_0\cosh\zeta x_1)\cos\zeta x_2 d\zeta,$$

$$\bar{\bar{\sigma}}^{(a)}_{22} = -\frac{\mu K a}{2\pi}\int_0^\infty G(\zeta,\xi_1)\left[(\mu_0\cosh\mu_0 - 2\sinh\mu_0)\sinh\zeta x_1 -\right.$$

$$\left.- \zeta x_1 \sinh\mu_0\cosh\zeta x_1\right]\cos\zeta x_2 d\zeta,$$

$$\bar{\bar{\sigma}}^{(a)}_{12} = -\frac{\mu K a}{2\pi}\int_0^\infty G(\zeta,\xi_1)\left[(\mu_0\cosh\mu_0 - \sinh\mu_0)\cosh\zeta x_1 -\right.$$

$$\left.- \zeta x_1 \sinh\mu_0\sinh\zeta x_1\right]\sin\zeta x_2 d\zeta,$$

where

$$G(\zeta,\xi_1) = -\frac{\mu_0\rho^{(a)}(\zeta,\xi_1)}{\sinh 2\mu_0 - 2\mu_0}.$$

We observe that for the arrangement which is symmetric with respect to the x_2-axis (Cf. Fig. 8.2a) the distribution of the normal stresses is symmetric with respect to the axes x_1 and x_2 the tangential stresses being antisymmetric. For the sources situated antisymmetrically with respect to the x -axis the converse statement is true. We note also that the state of stress $\bar{\bar{\sigma}}^{(a)}_{ij}$ vanishes for a heat source situated at the origin of the coordinate system.

The functions describing the state of stress $\bar{\bar{\sigma}}_{ij}$ are regular, and the singularities due to the action of the concentrated heat source appear in the formulae for $\bar{\sigma}_{ij}$ only.

For a source situated at the point $(\xi_1, 0)$ the final state of stress has the form

$$\sigma_{ij} = \bar{\sigma}_{ij} + \bar{\bar{\sigma}}_{ij}^{(s)} + \bar{\bar{\sigma}}_{ij}^{(a)}. \tag{8.26}$$

The temperature field is to be determined from equation (8.2):

$$T = \frac{1}{m_0}\nabla_1^2\phi = -\frac{Q_0}{2\mu K\kappa}(\bar{\sigma}_{1'1'} + \bar{\sigma}_{22}) = -\frac{Q_0}{\kappa}\vartheta(x_1', x_2; \xi_1', 0), \quad K = \frac{m_0 Q_0}{\kappa}, \tag{8.27}$$

where the function ϑ is given by relation (8.12). The thermal boundary conditions are satisfied, namely $T = 0$ for $x_1' = 0$ and $x_1' = a$. In the vicinity of the point $(\xi_1', 0)$ the temperature has a logarithmic singularity.

If the unit source be placed at the point (ξ_1', ξ_2), then in all formulae x_2 should be replaced by $x_2 - \xi_2$, and we set $Q_0 = 1$. Thus we obtain the Green function for the stresses and the temperature for sources distributed arbitrarily over the region of the plate.

9. Thermal stresses in infinite cylinders. Quasi-static problem.

The method of solution consists here in introducing the thermoelastic displacement potential and constructing the solution as a sum of two parts, the first part being connected with the potential ϕ and the second representing the residual solution by means of the Airy function F.

We determine the temperature from the equation

$$(\nabla_1^2 - \frac{1}{\kappa}\partial_t)T = -\frac{Q}{\kappa}. \tag{9.1}$$

We then determine the particular solution of the equation

$$\nabla_1^2\phi = mT. \tag{9.2}$$

Next we determine the displacements $\bar{u}_i$ and the stresses $\bar{\sigma}_{ij}$ corresponding to the function ϕ. In general the boundary conditions are not satisfied; thus for a stress-free boundary $\sigma_{nn} \neq 0$, $\sigma_{ns} \neq 0$. Hence the state of stress $\bar{\sigma}_{ij}$ has now to be completed by state $\bar{\bar{\sigma}}_{ij}$ expressed by Airy function F, such that on the boundary

$$\bar{\sigma}_{nn} + \bar{\bar{\sigma}}_{nn} = 0, \quad \bar{\sigma}_{ns} + \bar{\bar{\sigma}}_{ns} = 0. \tag{9.3}$$

The two following simple examples will illustrate the procedure.

Let a solid circular cylinder of radius a be situated in the initial temperature field T_0 and suppose that at the instant $t = 0$ its surface is cooled to the temperature $T = 0$ [112].

We have to solve equation (9.1) with the boundary condition

$$T(a,t) = 0 \tag{9.4}$$

and the initial temperature

$$T(r,0) = T_0, \qquad T_0 = \text{const.} \tag{9.5}$$

Taking the Laplace transform of equation (9.1) we obtain the equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\bar{T} - \frac{1}{\kappa}(p\bar{T} - T(r,0)) = 0, \tag{9.6}$$

with homogeneous boundary conditions and the initial condition (9.5). We have from (9.6):

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{p}{\kappa}\right)\bar{T}(r,p) = -\frac{1}{\kappa}T_0. \tag{9.7}$$

Introducing the finite Hankel transform

$$\tilde{T}(\alpha_n,p) = \int_0^a \overline{T}(r,p)rJ_0(\alpha_n r)\mathrm{d}r,$$
$$\overline{T}(r,p) = \frac{2}{a^2}\sum_{n=1}^{\infty} \tilde{T}(\alpha_n,p)\frac{J_0(\alpha_n r)}{|J_1(\alpha_n a)|^2} \tag{9.8}$$

We perform the integration by parts

$$\int_0^a \left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)r\overline{T}(r,p)J_0(\alpha_n r)\mathrm{d}r = \left[r\frac{\mathrm{d}\overline{T}}{\mathrm{d}r}J_0(\alpha_n r) - \alpha_n r\overline{T}J_0(\alpha_n r)\right]_0^a - $$
$$- \alpha_n^2\int_0^a r\overline{T}(r,p)J_0(\alpha_n r)\mathrm{d}r. \tag{9.9}$$

The expression in brackets vanishes for the upper limit, provided $J_0(\alpha_n a) = 0$; for $r = 0$ it vanishes also. The parameter α_n is taken to satisfy the transcendental equation

$$J_0(\alpha_n a) = 0, \quad n = 1,2,\ldots,\infty. \tag{9.10}$$

Integrating the equation (9.7) in the interval $\langle 0,a\rangle$ we have

$$\tilde{T}(\alpha_n,p) = \frac{T_0 a}{\kappa\alpha_n}\frac{J_1(\alpha_n a)}{\alpha_n^2+p/\kappa}, \tag{9.11}$$

Performing the inverse Hankel transformation we find that

$$\overline{T}(r,p) = 2T_0\sum_{n=1}^{\infty}\frac{J_0(\alpha_n r)}{a\alpha_n J_1(\alpha_n a)(\alpha_n^2+p/\kappa)}. \tag{9.12}$$

Applying the inverse Laplace transformation, we arrive finally at the result

$$T(r,t) = 2T_0\sum_{n=1}^{\infty}\frac{J_0(\alpha_n r)}{\alpha_n a J_1(\alpha_n a)}\exp(-\alpha_n^2\kappa t). \tag{9.13}$$

The particular solution of equation (9.2) can be represented by the series

$$\phi = -2T_0 m \sum_{n=1}^{\infty} \frac{J_0(\alpha_n r)}{\alpha_n^3 a J_1(\alpha_n a)} \exp(-\alpha_n^2 \kappa t). \qquad (9.14)$$

We have used the result

$$\int_0^a \left(\frac{d}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\overline{\phi}(r,p) r J_0(\alpha_n r) dr = -\alpha_n^2 \tilde{\phi}(\alpha_n, p).$$

Hence, we calculate

$$\overline{\sigma}_{rr} = -\frac{2\mu}{r}\frac{\partial\phi}{\partial r} = -4\mu T_0 m \sum_{n=1}^{\infty} \frac{\exp(-\alpha_n^2\kappa t)}{\alpha_n a J_1(\alpha_n a)} \frac{J_1(\alpha_n r)}{\alpha_n r},$$

$$\overline{\sigma}_{\phi\phi} = -2\mu\frac{\partial^2\phi}{\partial r^2} = -4\mu T_0 m \sum_{n=1}^{\infty} \frac{\exp(-\alpha_n^2\kappa t)}{\alpha_n a J_1(\alpha_n a)} \left(J_0(\alpha_n r) - \frac{J_1(\alpha_n r)}{\alpha_n r}\right),$$

$$\overline{\sigma}_{zz} = -2\mu\nabla_1^2\phi = -2\mu m T. \qquad (9.15)$$

It is readily observed that the stress $\overline{\sigma}_{rr}(a,t)$ does not vanish on the surface $r = a$ at the cylinder. To eliminate this stress we add to the state $\overline{\sigma}_{ij}$ a state $\overline{\overline{\sigma}}_{ij}$ in accordance with the relation

$$\overline{\overline{\sigma}}_{rr}(r,t) = \overline{\overline{\sigma}}_{\phi\phi}(r,t) = -\overline{\sigma}_{rr}(a,t) = 4\mu m T_0 \sum_{n=1}^{\infty} \frac{\exp(-\alpha_n^2\kappa t)}{(\alpha_n a)^2}, \qquad (9.16)$$

$$\overline{\overline{\sigma}}_{zz} = \nu(\overline{\overline{\sigma}}_{rr} + \overline{\overline{\sigma}}_{\phi\phi}).$$

Finally we have [112]

$$\sigma_{rr} = -4\mu T_0 m \sum_{n=1}^{\infty} \frac{\exp(-\alpha_n^2\kappa t)}{\alpha_n a J_1(\alpha_n a)} \left(\frac{J_1(\alpha_n r)}{\alpha_n r} - \frac{J_1(\alpha_n a)}{\alpha_n a}\right),$$

$$\sigma_{\phi\phi} = -4\mu T_0 m \sum_{n=1}^{\infty} \frac{\exp(-\alpha_n^2\kappa t)}{\alpha_n a J_1(\alpha_n a)} \left(J_0(\alpha_n r) - \frac{J_1(\alpha_n r)}{\alpha_n r} - \frac{J_1(\alpha_n a)}{\alpha_n a}\right) \qquad (9.17)$$

$$\sigma_{zz} = -4\mu T_0 m \sum_{n=1}^{\infty} \frac{\exp(-\alpha_n^2 \kappa t)}{\alpha_n a J_1(\alpha_n a)} \left(J_0(\alpha_n r) - 2\nu \frac{J_1(\alpha_n a)}{\alpha_n a} \right).$$

We next examine an infinite cylinder of rectangular cross-section $|x_1| < a/_2$, $|x_2| < b/_2$, the initial condition having the form

$$\left[T(x_1, x_2, t) \right]_{t=0} = T_0 = \text{const.} \tag{9.18}$$

Applying the finite cosine transformation, the temperature T and the function ϕ are given by the formulae [77]

$$T = \frac{16T_0}{ab} \sum_{n,m}^{\infty} \frac{(-1)^{(n+m-2)/2}}{\alpha_n \beta_m} \exp\left[-(\alpha_n^2 + \beta_m^2)\kappa t \right] \cos\alpha_n x_1 \cos\beta_m x_2,$$

$$\phi = -\frac{16mT_0}{ab} \sum_{n,m}^{\infty} \frac{(-1)^{(n+m-2)/2}}{\alpha_n \beta_m (\alpha_n^2 + \beta_m^2)} \exp\left[-(\alpha_n^2 + \beta_m^2)\kappa t \right] \cos\alpha_n x_1 \cos\beta_m x_2, \tag{9.19}$$

where $\alpha_n = \frac{n\pi}{a}$, $\beta_m = \frac{m\pi}{b}$ $(n,m = 1,3,5,\ldots,\infty)$ and a,b are the sides of the rectangle. Thus the stresses have the form

$$\sigma_{ij} = (\partial_i \partial_j - \delta_{ij} \nabla_1^2)(2\mu\phi - F) = \bar{\sigma}_{ij} + \bar{\bar{\sigma}}_{ij}, \quad i,j = 1,2,$$
$$\sigma_{33} = -\nabla_1^2 (2\mu\phi - \nu F), \tag{9.20}$$

where

$$F = \sum_{m=1,3,.}^{\infty} \beta_m^{-2} (A_m \cosh\beta_m x_1 + B_m \beta_m \sinh\beta_m x_1) \cos\beta_m x_2 +$$
$$+ \sum_{m=1,3,..}^{\infty} \alpha_n^{-2} (C_n \cosh\alpha_n x_2 + D_n \alpha_n x_2 \sinh\alpha_n x_1) \cos\alpha_n x_1 \tag{9.21}$$

and the constants $A_m, \ldots, D_n$ are determined from the boundary conditions

$$\bar{\sigma}_{11} + F_{,22} = 0, \quad \bar{\sigma}_{12} - F_{,12} = 0 \text{ for } x_1 = a/2,$$
$$\bar{\sigma}_{22} + F_{,11} = 0, \quad \bar{\sigma}_{12} - F_{,12} = 0 \text{ for } x_2 = b/2. \tag{9.22}$$

Consequently, we obtain the system of equations

$$A_m\eta(\mu_m) + \frac{16}{b^2}\beta_m^{\,2}(-1)^{\frac{m-1}{2}}\sum_{n=1,3,\ldots}^{\infty} C_n\frac{(-1)^{(n-1)/2}\cosh^2\delta_n}{(\alpha_n^{\,2}+\beta_m^{\,2})^2\sinh\delta_n} =$$
$$= -\frac{32\mu m T_0}{ab}(-1)^{\frac{m-1}{2}}e^{-\beta_m^{\,2}\kappa t}\sum_{n=1,3,\ldots}^{\infty}\frac{e^{-\alpha_n^{\,2}\kappa t}}{\alpha_n^{\,2}+\beta_m^{\,2}}, \tag{9.23}$$

$$C_n\eta(\delta_n) + \frac{16}{a^2}\alpha_n^{\,2}(-1)^{\frac{n-1}{2}}\sum_{m=1,3,\ldots}^{\infty} A_m\frac{(-1)^{\frac{m-1}{2}}\cosh^2\mu_m}{(\alpha_n^{\,2}+\beta_m^{\,2})\sinh\mu_m} =$$
$$= -\frac{32\mu m T_0}{ab}(-1)^{\frac{n-1}{2}}e^{-\alpha_n^{\,2}\kappa t}\sum_{m=1,3,\ldots}^{\infty}\frac{e^{-\beta_m^{\,2}\kappa t}}{\alpha_n^{\,2}+\beta_m^{\,2}},$$

$$B_m = -A_m\mu_m^{-1}\cot\mu_m, \quad D_n = -C_n\delta_n^{-1}\cot\delta_n,$$

where

$$\eta(\mu_m) = \frac{2\mu_m+\sinh 2\mu_m}{2\mu_m\sinh\mu_m}, \qquad \eta(\delta_n) = \frac{2\delta_n+\sinh 2\delta_n}{2\delta_n\sinh\delta_n},$$
$$\mu_m = \frac{\beta_m a}{2}, \qquad \delta_n = \frac{\alpha_n b}{2}.$$

The constants $A_m, \ldots, D_n$ being known the function F is also known, and hence we can calculate the stresses.

10. The linear theory of dislocations. The Volterra concept of dislocations.

We know that a classical dislocation line L is associated with a

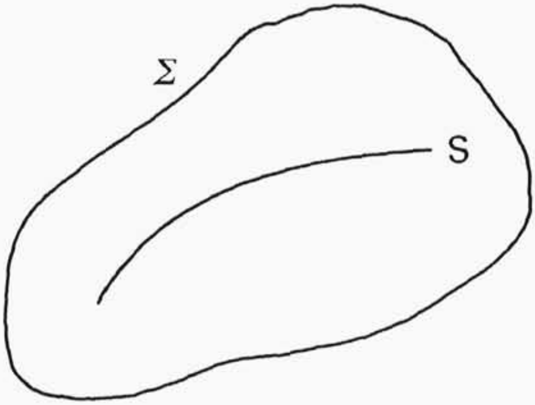

Fig. 10.1

displacement field $\underset{\sim}{u}(\underset{\sim}{x})$ which is discontinuous on the surface S immersed in an elastic solid, and the jump of $u_l(\underset{\sim}{x})$ on S, the well-known Burgers' vector b_l, is given by the formula

$$|u_l(\underset{\sim}{x})| = u_l^+(\underset{\sim}{x}) - u_l^-(\underset{\sim}{x}) = b_l \tag{10.1}$$

We shall define a Volterra dislocation [120] in an elastic solid as a displacement $\underset{\sim}{u}(\underset{\sim}{x})$ with the following properties:

a) $\underset{\sim}{u}$ is a smooth field throughout the solid, except for S.

b) $\underset{\sim}{u}$ has a jump on S given by (10.1).

c) partial derivatives of $\underset{\sim}{u}$ are continuous across S.

d) the displacement $u(\underset{\sim}{x})$ satisfies, outside of S, the following equations of elasticity:-

$$\mu\nabla^2 u_i + (\lambda+\mu)u_{j,ji} = 0 \tag{10.2}$$

or

$$A_{ijkl}u_{k,lj} = 0 \tag{10.3}$$

where

$$A_{ijkl} = \mu(\delta_{jl}\delta_{ik}+\delta_{il}\delta_{jk})+\lambda\delta_{ij}\delta_{kl}.$$

In what follows, an important role will be played by the Green's function for an infinite elastic solid.

Suppose that V is an infinite domain of three-dimensional euclidean space, and that a concentrated force $X_i^{(n)} = \delta_{in}\delta(\underset{\sim}{x}-\underset{\sim}{x}')$ is applied at $x' \in V$ parallel to the x_n-axis.

Then the displacement field $G_{in}(\underset{\sim}{x},\underset{\sim}{x}')$ satisfies the following system of equations

$$A_{ijkl}G_{kn,lj}+\delta_{in}\delta(\underset{\sim}{x}-\underset{\sim}{x}') = 0. \tag{10.4}$$

The solution of (10.4) for the infinite space is well-known; it takes the form [101]

$$G_{in} = \frac{1}{8\pi\mu}(\delta_{in}R_{,pp} - \frac{\lambda+\mu}{\lambda+2\mu}R_{,in}), \tag{10.5}$$

where

$$R = \left[(x_i-x_i')(x_i-x_i')\right]^{\frac{1}{2}}.$$

Assume now that V is a simply connected body with the boundary Σ shown in the Fig. 10.1. Since $\underset{\sim}{u}$ is continuous on V,

$$u_n(\underset{\sim}{x}') = \int_V u_i(\underset{\sim}{x})\delta_{in}\delta(\underset{\sim}{x}-\underset{\sim}{x}')dV(\underset{\sim}{x}) = -A_{ijkl}\int_V u_i(\underset{\sim}{x})G_{kn,lj}dV(\underset{\sim}{x})$$

$$= -A_{ijkl}\int_\Sigma\left(u_iG_{kn,l}d\Sigma_j-u_{i,j}G_{kn}d\Sigma_l\right)-A_{ijkl}\int_V u_{i,jl}G_{kn}dV. \tag{10.6}$$

But $A_{ijkl}G_{kn}u_{i,jl} = G_{kn}A_{kjil}u_{i,jl} = 0$ in view of (10.3). If $\Sigma \to S$ then in view of the condition $|\underset{\sim}{u}| = \underset{\sim}{b}$ on S, we obtain

$$u_n(\underset{\sim}{x}') = -b_i \int_S A_{ijkl} \frac{\partial}{\partial \xi_l} G_{kn}(\underset{\sim}{x}', \underset{\sim}{\xi}) dS_j(\underset{\sim}{\xi}). \tag{10.7}$$

Consider the latter integral

$$u_n(\underset{\sim}{x}') = -b_i A_{ijkl} \int_S \frac{\partial}{\partial \xi_l} G_{kn}(\underset{\sim}{x}', \underset{\sim}{\xi}) dS_j(\underset{\sim}{\xi}) = b_i A_{ijkl} \int_S \frac{\partial}{\partial x_l'} G_{kn}(\underset{\sim}{x}', \underset{\sim}{\xi}) dS_j(\underset{\sim}{\xi})$$

$$= b_i A_{ijkl} \int_v dV(\underset{\sim}{x}) \delta(\underset{\sim}{x} - \underset{\sim}{\xi}) \frac{\partial}{\partial x_l'} G_{kn}(\underset{\sim}{x}', \underset{\sim}{x}) \int_S dS_j(\underset{\sim}{\xi})$$

$$= -b_i A_{ijkl} \int_v dV(\underset{\sim}{x}) \frac{\partial}{\partial x_l} G_{kn}(\underset{\sim}{x}', \underset{\sim}{x}) \int_S \delta(\underset{\sim}{x} - \underset{\sim}{\xi}) dS_j(\underset{\sim}{\xi})$$

$$= b_i A_{ijkl} \int_v dV(\underset{\sim}{x}) G_{kn}(\underset{\sim}{x}', \underset{\sim}{x}) \frac{\partial}{\partial x_l} \int_S \delta(\underset{\sim}{x} - \underset{\sim}{\xi}) dS_j(\underset{\sim}{\xi}).$$

Introducing the notation

$$\eta_{ji} = b_i \int_S \delta(\underset{\sim}{x} - \underset{\sim}{\xi}) dS_j(\underset{\sim}{\xi}), \tag{10.8}$$

we can rewrite the formula (10.7) in the form [52]

$$u_n(\underset{\sim}{x}') = \int_v dV(\underset{\sim}{x}) \left(A_{ijkl} G_{kn}(\underset{\sim}{x}, \underset{\sim}{x}') \frac{\partial}{\partial x_l} \eta_{jl} \right). \tag{10.9}$$

Now we replace the discontinuous displacement field (with the discontinuity on S) by the displacement field related to the fictitious body forces X_i.

Accordingly, instead of (10.3) we should consider the following non-homogeneous system of equations

$$A_{ijkl} u_{k,jl} + X_i = 0. \tag{10.10}$$

A solution of (10.11) can be obtained in terms of the Green functions G_{in}, using the following reciprocal theorem for the infinite elastic space

$$\int_v X_i u_i' \mathrm{d}V = \int_v X_i' u_i \mathrm{d}V. \tag{10.11}$$

Indeed, substituting

$$\{u_i', X_i'\} = \{G_{in}, \quad \delta(\underset{\sim}{x}-\underset{\sim}{x}')\delta_{in}\}$$

into (10.11), we obtain

$$u_n(\underset{\sim}{x}') = \int_v X_i G_{in} \mathrm{d}V(\underset{\sim}{x}). \tag{10.12}$$

Now if the dislocation (10.9) is to be identical with (10.12) we have to assume that

$$X_k = A_{ijkl}\eta_{ji,l} = b_i A_{ijkl}\frac{\partial}{\partial x_l}\int_S \delta(\underset{\sim}{x}-\underset{\sim}{\xi})\mathrm{d}S_j(\underset{\sim}{\xi}), \tag{10.13}$$

and the dislocation $|\underset{\sim}{u}|$ characterised by the Burgers' vector $\underset{\sim}{b}$ can also be described by the field of body forces related to $\underset{\sim}{b}$ through (10.13).

We can show the transition from residual distortions to the dislocation. In section (1.1) we have obtained the equations (1.6)

$$\mu\nabla^2 u_i + (\lambda+\mu)u_{j,ji} + \hat{X}_i = 0 \tag{10.14}$$

or

$$A_{ijkl}u_{k,jl} + \hat{X}_i = 0, \quad \hat{X}_i = \sigma^0_{ji,j}. \tag{10.15}$$

Using the Green functions G_{in} we have

$$u_n(\underset{\sim}{x}') = \int_v \hat{X}_k(\underset{\sim}{x})G_{kn}(\underset{\sim}{x},\underset{\sim}{x}')\mathrm{d}V(\underset{\sim}{x}). \tag{10.16}$$

But

$$\hat{X}_k = -\hat{\sigma}_{jk,j} = -A_{ijkl}\overset{0}{\varepsilon}_{ij,l} \qquad (10.17)$$

Comparing now (10.16) with (10.12), we conclude that the transition from the residual distortions $\overset{0}{\varepsilon}_{ij}$ to the Volterra dislocations can be expressed by the relations

$$\overset{0}{\varepsilon}_{ij} = -\overset{0}{\eta}_{ij} = -b_j\int_v \delta(\underset{\sim}{x}-\underset{\sim}{\xi})\mathrm{d}S_i(\underset{\sim}{\xi}). \qquad (10.18)$$

Substituting from (10.18) into (10.16) we obtain

$$u_n(\underset{\sim}{x}') = -\int_v A_{ijkl}\overset{0}{\varepsilon}_{ji,l}\, G_{kn}\mathrm{d}V(\underset{\sim}{x}) \qquad (10.19)$$

$$= \int_v \overset{0}{\varepsilon}_{ji}A_{ijkl}\, G_{kn,l}\mathrm{d}V(\underset{\sim}{x})$$

$$= \int_v \overset{0}{\varepsilon}_{ji}(\underset{\sim}{x})\sigma_{ji}^{(n)}(\underset{\sim}{x},\underset{\sim}{x}')\mathrm{d}V(\underset{\sim}{x}).$$

An explicit form of u is obtained if $\sigma_{ji}^{(n)}$ are expressed by the Green tensor G_{in}. The final result is

$$u_k(\underset{\sim}{x}') = \frac{1}{8\pi}\int_v (\overset{0}{\varepsilon}_{jk}R_{,jpp}+\overset{0}{\varepsilon}_{kj}R_{,ppj}-\frac{1}{1-\nu}R_{,kpj}\overset{0}{\varepsilon}_{jp}+\frac{1}{1-\nu}R_{,kpp}\overset{0}{\varepsilon}_{rr})\mathrm{d}V(\underset{\sim}{x}). \qquad (10.20)$$

Formulae (10.20) can be used to discuss particular types of dislocations.

The edge dislocation is characterized by the Burgers' vector of the form $\underset{\sim}{b} = (b_1,0,0)$. If the dislocation axis coincides with the x_3-axis and the dislocation surface is the semi-plane x_1x_3 with x_1 negative, then the only non-vanishing component of $\overset{0}{\varepsilon}_{ij}$ by virtue of (10.18) is given by

$$\varepsilon_{21}^{0} = -b_1\int_S \delta(\underset{\sim}{x}-\underset{\sim}{\xi})dS_2(\underset{\sim}{\xi}) = b_1\int_{-\infty}^{0}\delta(x_1-\xi_1)d\xi_1\delta(x_2)\int_{-\infty}^{\infty}\delta(x_3-\xi_3)d\xi_3 \qquad (10.21)$$

$$\varepsilon_{21}^{0} = b_1 H(-x_1)\delta(x_2), \qquad dS_2 = -d\xi_1 d\xi_2,$$

where $H(-x_1)$ is the Heaviside function defined by

$$H(-x_1) = \begin{cases} 0 \text{ for } x_1 > 0 \\ 1 \text{ for } x_1 < 0. \end{cases}$$

Substituting (10.21) into (10.20) after carrying out integrations, we obtain

$$u_1(x_1,x_2) = \frac{b_1}{2\pi}\left(\operatorname{arctg}\frac{x_2}{x_1} + \frac{\lambda+\mu}{\lambda+2\mu}\frac{x_1x_2}{r^2}\right),$$

$$u_2(x_1,x_2) = \frac{b_1}{2\pi}\left(\frac{\mu}{\lambda+2\mu}\ln r + \frac{\lambda+\mu}{\lambda+2\mu}\frac{x_2^2}{r^2}\right). \qquad (10.22)$$

We can obtain the stresses σ_{ij}, using the compatibility equation

$$\nabla_1^2\nabla_1^2 F = \frac{4\mu(\lambda+\mu)}{\lambda+2\mu}A \qquad (10.23)$$

where

$$A = -\partial_2^2\varepsilon_{11}^0 - \partial_1^2\varepsilon_{22}^0 + \partial_1\partial_2(\varepsilon_{12}^0+\varepsilon_{21}^0) = -b_1\delta(x_1)\partial_2\delta(x_2).$$

The solution of equation (10.23) will be obtained by applying the double exponential Fourier transformation.

In this way we get

$$\hat{F} = -\eta b_1\frac{i\xi_2}{(\xi_1^2+\xi_2^2)^2}, \qquad \eta = \frac{4\mu(\lambda+\mu)}{\pi(\lambda+2\mu)}. \qquad (10.24)$$

Using the Fourier inversion theorem, we obtain

$$F = -\frac{\eta b_1}{2\pi}\frac{\partial}{\partial x_2}\iint_{-\infty}^{\infty}\frac{\exp\left[-i(\xi_1 x_1+\xi_2 x_2)\right]d\xi_1 d\xi_2}{(\xi_1^2+\xi_2^2)^2}. \tag{10.25}$$

The above integral does not exist as an improper integral; nor can we assign to it a principal Cauchy value. We can however separate out of it the "finite part" [44].

We obtain the "finite part" in the form

$$\frac{1}{2\pi}\iint_{-\infty}^{\infty}\frac{\exp\left[-i(\xi_1 x_1+\xi_2 x_2)\right]d\xi_1 d\xi_2}{(\xi_1^2+\xi_2^2)^2} = \frac{r^2}{4}(C+\ln r) \tag{10.26}$$

and

$$F = -\frac{\mu(\lambda+\mu)}{2\pi(\lambda+2\mu)}\frac{\partial}{\partial x_2}\left[r^2(\ln r+C)\right]. \tag{10.27}$$

Making use of the formulae

$$\sigma_{\alpha\beta} = (\delta_{\alpha\beta}\nabla_1^2-\partial_\alpha\partial_\beta)F, \qquad \alpha,\beta = 1,2,$$

we obtain the stresses

$$\begin{aligned}
\sigma_{11} &= -\frac{\mu b_1}{2\pi(1-\nu)}\left(\frac{x_2}{r^2}+\frac{2x_1^2x_2}{r^4}\right),\\
\sigma_{22} &= -\frac{\mu b_1}{2\pi(1-\nu)}\left(\frac{x_2}{r^2}-\frac{2x_1^2x_2}{r^4}\right),\\
\sigma_{12} &= \frac{\mu b_1}{2\pi(1-\nu)}\left(\frac{x_1}{r^2}-\frac{2x_1x_2^2}{r^4}\right), \qquad \nu = \frac{\lambda}{2(\lambda+\mu)}.
\end{aligned} \tag{10.28}$$