Rough Probability
and Partial
observability

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ROUGH PROBABILITY AND PARTIAL OBSERVABILITY

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Abstract - Содержание - Streszczenie

In this paper we define the notion of partial observability considered in statistical models in terms of "rough" sets. With each event we associate "inner" and "outer" probability, or an interval which end points are inner and outer probability respectively. The interval is called rough probability of the event. Some elementary properties of rough probabilities are given.

Приближенное множество и частичная наблюдаемость

В работе определено понятие частичной наблюдаемости, рассмотряемой в статистических моделях в терминах приближенных множеств. С каждой произвольной связаны "внутренняя" и "внешняя" вероятности или интервалы, конечными являются внутренняя и внешняя вероятности. Этот интервал называется приближенной вероятностью. Представлены элементарные свойства приближенных вероятностей.

Zbiory przybliżone i częściowa obserwowałność

W pracy zdefiniowano pojęcie częściowej obserwowałności rozważanej w modelach statystycznych w terminach zbiorów przybliżonych. Z każdym zdarzeniem związane jest "wewnętrzne" i "zewnętrzne" prawdopodobieństwo lub przedział, którego końcami są prawdopodobieństwo wewnętrzne i zewnętrzne. Przedział ten jest nazywany prawdopodobieństwem przybliżonym. Podano elementarne własności prawdopodobieństw przybliżonych.
1. Introduction

Fleszczynska and Dambrowska introduced partial observability in statistical models (see [1]).

We propose in this paper somewhat different formulation of this notion based on the concept of the rough set (see [2]) - which can be of interest in situations when the "exact" probability of some events is not known, but only the interval, to which this probability belongs.

As a departure point of our considerations we introduce the notion of a stochastic approximation space $S$ and the inner and outer probability $P_S(X), \overline{P}_S(X)$ of the event $X$ in the approximation space $S$ is defined. In fact the inner and outer probability of an event $X$ is the probability of the interior and closure of $X$ respectively, in the topological space generated by the approximation space $S$. Some elementary properties of inner and outer probabilities are given and the notion of the rough (approximate) probability of an event $X$ is defined, as $P^*_S(X) = \langle P_S(X), \overline{P}_S(X) \rangle$, which is to understand as an interval to which the probability of $X$ belongs. Elementary properties of rough probability are given.

We consider finite stochastic approximation spaces only, but the infinite case can be also treated in a similar way.

We assume that the reader is familiar with the basic notions of set theory and topology, and we use standard mathematical notation throughout the paper.
2. Approximation space, approximation of sets

In this section we recall after [1] basic notions concerning the concept of a rough set, used as a departure point of our considerations.

Let \( U \) be a certain set, and let \( R \) be an equivalence relation on \( U \). The pair \( A = (U, R) \) will be called an approximation space, and \( R \) will be referred to as an indiscernibility relation. If \( x, y \in U \) and \( R(x, y) \) we say that \( x \) and \( y \) are indistinguishable in \( A \).

Equivalence classes of the relation \( R \) and the empty set \( \emptyset \) will be called elementary sets (atoms) in \( A \) or in short elementary sets (atoms) if \( A \) is understood.

Every union of elementary sets in \( A \) will be called a composed set in \( A \), or in short a composed set, if \( A \) is known.

The family of all composed sets in \( A \) is denoted by \( \text{Com}(A) \).

Obviously \( \text{Com}(A) \) is a Boolean algebra, i.e., the family of all composed sets is closed under intersection, union and complement of sets.

Let \( X \) be a certain subset of \( U \). The least composed set in \( A \) containing \( X \) will be called the best upper approximation of \( X \) in \( A \), in symbols \( \overline{A(X)} \); the greatest composed set in \( A \) contained in \( X \) will be called the best lower approximation of \( X \) in \( A \), and will be denoted by \( \underline{A(X)} \).

3. Properties of approximations

One can easily check that the approximation space \( A = (U, R) \) defines uniquely the topological space \( T_A = (U, \text{Com}(A)) \), and \( \text{Com}(A) \) is the family of all open and closed sets in \( T_A \), and \( U/R \) is a base for \( T_A \).

From the definition of approximations follows that \( \overline{A(X)} \) and \( \underline{A(X)} \) are interior and closure of \( X \) in the topological space \( T_A \) respectively.

Thus for every \( X \subseteq U \) and every approximation space \( A = (U, R) \) the following properties of approximations are valid:

\[
\begin{align*}
& (A1) \quad \overline{A(X)} \subseteq X \subseteq \underline{A(X)} \\
& (A2) \quad \overline{A(U)} = \underline{A(U)} = U \\
& (A3) \quad \overline{A(\emptyset)} = \underline{A(\emptyset)} = \emptyset \\
& (A4) \quad \overline{A(X)} \cup \overline{A(Y)} = \overline{A(X \cup Y)} \\
& (A5) \quad \overline{A(X)} = \overline{\overline{A(X)}} = A(X) \\
& (A6) \quad \overline{A(X \cap Y)} = \overline{A(X)} \cap \overline{A(Y)} \\
& (A7) \quad \overline{A(X \cap Y)} = \overline{A(X)} \cap \overline{A(Y)} \\
& (A8) \quad \overline{A(X \cup Y)} = \overline{A(X)} \cup \overline{A(Y)} \\
& (A9) \quad \overline{A(\neg X)} = \overline{A(X)} \\
& (A10) \quad \overline{A(\neg X)} = \overline{A(X)} \\
& (A11) \quad \overline{A(\neg X)} = \overline{A(X)} \\
& (A12) \quad \overline{A(X)} \subset \overline{A(Y)} \text{ and } \underline{A(X)} \subset \underline{A(Y)} \\
& \text{Moreover we have the property} \\
& (A13) \quad \overline{A(X)} \supset \overline{A'(X)} \\
& (A14) \quad \overline{A(X)} \subseteq \overline{A'(X)} \\
& \text{for every } X \subseteq U.
\end{align*}
\]
4. Observable and unobservable sets

Let \( A = \{U, R\} \) be an approximation space.

If \( X \subseteq U \) and \( \overline{A}(X) = U \) we say that \( X \) is dense in \( A \).

If \( X \subseteq U \) and \( \overline{A}(X) = \emptyset \) we say that \( X \) is co-dense in \( A \).

If \( X \subseteq U \) and \( X \) is both dense and co-dense in \( A \) we say that \( X \) is dispersed in \( A \).

If we assume that we are able to observe only elementary sets and their unions, i.e., composed sets, then we can classify subsets of the approximation space \( A = \{U, R\} \) in the following way:

1) If \( \overline{A}(X) = \overline{A}(X) \) then \( X \) will be called observable in \( A \), otherwise set \( X \) is unobservable in \( A \).

Let \( X \subseteq U \) and let \( X \) be unobservable in \( A \). We introduce the following four categories of unobservable sets:

2) If \( A(X) \neq \emptyset \) and \( A(X) \neq U \), then we shall call \( X \) roughly observable in \( A \).

3) If \( A(X) \neq \emptyset \) and \( A(X) = U \), set \( X \) will be called externally unobservable in \( A \).

4) If \( A(X) = \emptyset \) and \( A(X) \neq U \), set \( X \) will be called internally unobservable in \( A \).

5) If \( A(X) = \emptyset \) and \( A(X) = U \), then \( X \) will be called totally unobservable in \( A \).

Let us now give an intuitive motivation for the above classification.

The notion of an observable set is obvious. As mentioned above if we are able to observe some elementary sets then the only observable sets are elementary sets and their unions.

If set \( X \) is roughly observable in \( A \), that is to mean that we are able to observe set \( X \) with a certain approximation only, i.e., to observe only lower and upper approximations of \( X \).

If set \( X \) is externally unobservable we are able to observe only lower approximations, because \( \overline{A}(X) = U \) means that we cannot exclude any element \( x \in U \) being possibly member of \( X \).

If set \( X \) is internally unobservable in \( A \) it means that we are unable to say for sure that any \( x \in U \) is a member of \( X \).

Finally, if set \( X \) is totally unobservable in \( A \) it means that we cannot exclude any element \( x \in U \) being possibly a member of \( X \) and we cannot also say for sure that any \( x \in U \) is a member of \( X \). Thus in fact we cannot observe set \( X \) in \( A \).

5. Example

Let us consider an information system as an example of an approximation space.

By an information system \( S \) (see [3]) we mean

\[
S = \langle U, A, V, \xi \rangle
\]

where

- \( U \) is a finite set of objects
- \( A \) is a finite set of attributes
- \( V = \bigcup \{V_a\}_{a \in A} \) is a finite set of values of attributes
- \( \xi : U \times A \rightarrow V \) is an information function such that \( \xi(x, a) \in V_a \) for every \( x \in U, a \in A \).

Instead the function \( \xi \) we shall use the function \( \xi_x \) such that \( \xi_x(a) = \xi(x, a) \) for every \( x \in U \) and \( a \in A \).

We say that objects \( x, y \in U \) are indiscernible in \( S \) if \( \xi_x = \xi_y \).

The indiscernibility relation generated by the information system \( S \) will be denoted by \( \sim \). Obviously \( S \) is an equivalence relation, so each information system \( S = \langle U, A, V, \xi \rangle \) generates an approximation space \( A_S = \langle U, \xi \rangle \).
Let us consider an information system with objects

\[ U = \langle x_0, \ldots, x_{11} \rangle, \text{ three attributes } A = \{ a, b, c \}, \text{ the sets of values } V_a = \langle u_1, u_2 \rangle, \ V_b = \langle v_1, v_2, v_3 \rangle \text{ and } V_c = \langle w_1, w_2, w_3 \rangle \text{ and the information function defined by the table} \]

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
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<tbody>
<tr>
<td>( x_0 )</td>
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<td>( x_4 )</td>
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<td>( x_5 )</td>
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<td>( x_6 )</td>
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<tr>
<td>( x_7 )</td>
<td>( u_1 )</td>
<td>( v_3 )</td>
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<tr>
<td>( x_8 )</td>
<td>( u_2 )</td>
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<tr>
<td>( x_9 )</td>
<td>( u_1 )</td>
<td>( v_2 )</td>
<td>( w_1 )</td>
</tr>
<tr>
<td>( x_{10} )</td>
<td>( u_1 )</td>
<td>( v_3 )</td>
<td>( w_2 )</td>
</tr>
</tbody>
</table>

Elementary sets (atoms) are in this system the following:

\[ E_1 = \langle x_0, x_1 \rangle \]
\[ E_2 = \langle x_2, x_6, x_9 \rangle \]
\[ E_3 = \langle x_3, x_5 \rangle \]
\[ E_4 = \langle x_4, x_8 \rangle \]
\[ E_5 = \langle x_7, x_{10} \rangle \]

Sets

\[ X_1 = \langle x_0, x_1, x_4, x_9 \rangle \]
\[ Y_1 = \langle x_3, x_5, x_7, x_{10} \rangle \]
\[ Z_1 = \langle x_2, x_3, x_5, x_6, x_9 \rangle \]

are observable in the information system.

Sets

\[ X_2 = \langle x_0, x_3, x_4, x_9, x_{10} \rangle \]
\[ Y_2 = \langle x_1, x_7, x_8, x_{10} \rangle \]
\[ Z_2 = \langle x_2, x_3, x_4, x_8 \rangle \]

are roughly observable in the system and the following are the corresponding approximations

\[ A(X_2) = \langle x_0, x_1, x_3, x_4, x_5, x_7, x_9, x_{10} \rangle = \bar{E}_4 \cup \bar{E}_5 \]
\[ A(Y_2) = \langle x_0, x_1, x_4, x_7, x_8, x_{10} \rangle = \bar{E}_2 \cup \bar{E}_3 \]
\[ A(Z_2) = \langle x_2, x_3, x_4, x_8 \rangle = \bar{E}_3 \]

Sets

\[ X_3 = \langle x_0, x_1, x_2, x_3, x_4, x_7 \rangle \]
\[ Y_3 = \langle x_1, x_2, x_3, x_6, x_9, x_{10} \rangle \]
\[ Z_3 = \langle x_0, x_2, x_3, x_4, x_9, x_{10} \rangle \]

are externally unobservable in the system.

Sets

\[ X_4 = \langle x_0, x_2, x_3 \rangle \]
\[ Y_4 = \langle x_1, x_2, x_4, x_7 \rangle \]
\[ Z_4 = \langle x_2, x_3, x_4 \rangle \]

are internally unobservable in the system.

Sets

\[ X_5 = \langle x_0, x_2, x_3, x_4, x_7 \rangle \]
\[ Y_5 = \langle x_1, x_5, x_6, x_9, x_{10} \rangle \]
\[ Z_5 = \langle x_0, x_2, x_5, x_7, x_9 \rangle \]
are totally unobservable in the system.

In other words observable sets can be exactly described by
means of attributes available in the information system. Roughly
observable sets cannot be described exactly by means of the at-
tributes of the information system. We can give only in this case
the best lower and upper description of the set by the attributes.
Extremely unobservable sets have only nontrivial lower ap-
proximation, and internally unobservable sets have only nontrivial
upper approximation (by trivial approximation we understand the
empty set \( \emptyset \) and the universe \( U \)). That is to mean that the set
of attributes available in the information system is not power-
ful enough to exclude any object being a member of the considered
set - in the first case and assert that some objects are for sure
members of the set. Totally unobservable sets are impossible to
describe by the attributes of the information system.

6. Stochastic approximation space, inner and outer probabi-
licity

The purpose of this section is to give probabilistic inter-
pretation of the notions given in previous sections.

Let \( A = (U,R) \) be a finite approximation space (i.e., space
where \( U \) is finite, any subset of \( U \) will be called an event
in \( A \). In particular one-element event is called primitive in
\( A \) and elementary sets in \( A \) are called elementary (or atomic)
events in \( A \). Observable sets in \( A \) are called observable events
in \( A \).

By a stochastic approximation space we mean an ordered
triple

\[ S = (U,R,P) \]

where \( A = (U,R) \) is an approximation space, \( (U,R,P) \) an underlying
space, and \( P \) is a probability measure defined on observable sets
in \( A \).

Evidently \( P(\emptyset) = 0, P(U) = 1 \), and if \( X = \bigcup_{i=1}^{n} X_i \) is an
observable set in \( A \) and \( X_i \) are atomic sets in \( A \), then
\[
P(X) = \sum_{i=1}^{n} P(X_i).
\]

Our aim is to evaluate probability of unobservable events.
(We recall that we are not given probabilities of primitive
events).

In order to investigate the problem we introduce inner and
outer probability of an event in the stochastic approximation
space \( (U,R,P) \), denoted as \( P_S(X) \) and \( P_S(X) \) respectively
and defined as follows:

\[
P_S(X) = P(A(X))
\]
\[
P_S(X) = P(A(X)),
\]

where \( A = (U,R) \) is the underlying approximation space of
\( S = (U,R,P) \).

From the definition of the probability measure and properties
of approximations we get the following properties of inner
and outer probabilities:

(B1) If \( X \) is observable in \( A \) then \( P_S(X) = P(X) = P_S(X) \)

(B2) \( P_S(X) \leq P(X) \leq P_S(X) \)

(B3) \( P_S(\emptyset) = P_S(\emptyset) = 0 \)

(B4) \( P_S(U) = P_S(U) = 1 \)

(B5) \( P_S(\neg X) = (1 - P_S(X)) \)

(B6) \( P_S(\neg X) = (1 - P_S(X)) \)
(B7) \( P_S(X \cup Y) \geq P_S(X) + P_S(Y) \) provided \( A(X) \cap A(Y) = \emptyset \)

(B8) \( P_S(X \cup Y) = P_S(X) + P_S(Y) \) provided \( A(A) \cap A(Y) = \emptyset \)

(B9) \( P_S(X \cap Y) = P_S(X) \cap P_S(Y) \) provided \( A(X) \) and \( A(Y) \) are stochastically independent

(B10) \( P_S(X \cap Y) \leq P_S(X) \cap P_S(Y) \) provided \( A(X) \) and \( A(Y) \) are stochastically independent.

In general case we have

(B11) \( P_S(X \cup Y) = P_S(X) + P_S(Y) - P_S(X \cap Y) \)

(B12) \( P_S(X \cup Y) = P_S(X) + P_S(Y) - P_S(X \cap Y) \)

Let \( S = (U,R,F) \) and \( S' = (U,R',F) \) be two stochastic approximation spaces; if \( A' = (U,R') \) is finer than \( A = (U,R) \), we shall say that also \( S' \) is finer than \( S \).

Obviously the following is true:

(B13) \( P_S(X) \geq P_S(X) \)

(B14) \( P_S(X) \leq P_S(X) \)

7. Rough probability

With every event \( X \) in a stochastic approximation space \( S = (U,R,F) \) we associate the interval \( P_S(X) \subset \langle 0, 1 \rangle \) defined as

\[
P_S(X) = \langle P_S(X) \rangle
\]

and \( P_S(X) \) will be called rough probability of \( X \) in \( S \).

Thus \( P_S(X) \) is the interval to which belongs the probability of the unobservable event \( X \).

Of course if \( X \) is an observable event in \( S \) then

\[
P_S(X) = P_S(X) = P(X)
\]

and

\[
P_S(X) = \langle P(X), P(X) \rangle
\]

or simple

\[
P_S(X) = P(X),
\]

i.e. \( P_S(X) \) reduces to one point.

Certainly \( P_S(X) \) has the following properties:

(C1) \( P_S(\emptyset) = 0 \)

(C2) \( P_S(U) = 1 \)

(C3) \( P_S(-X) = \langle 1 - P_S(X), 1 - P_S(X) \rangle \)

(C4) \( P_S(X \cup Y) \subset \langle P_S(X), P_S(Y), P_S(X) + P_S(Y), 1 \rangle \)

(C5) \( P_S(X \cap Y) \subset \langle P_S(X), P_S(Y), P_S(X), P_S(Y) \rangle \)

Obviously we have the following properties:

(a) If \( X \) is externally unobservable in \( A \), then

\[
P_S(X) = \langle P_S(X), 1 \rangle
\]

(b) If \( X \) is internally unobservable in \( A \) then

\[
P_S(X) = \langle 0, P_S(X) \rangle
\]

(c) If \( X \) is totally unobservable in \( S \), then

\[
P_S(X) = \langle 0, 1 \rangle
\]

In other words: if the event \( X \) is observable in \( A \), then we can give exact probability \( P(X) \); if \( X \) is roughly observable in \( A \), then we can give the interval \( P_S(X) \) to which the probability of event \( X \) belongs; if \( X \) is externally unobservable in \( A \), then we can give only lower bound of the probability of \( X \); if \( X \) is internally unobservable in \( A \), then we can give only
upper bound of the probability of $X_i$ if $X$ is totally unobservable in $A$ than we cannot give any bounds for the probability of $X$.

Moreover we have the following property:

If $S'$ is finer than $S$, then

$$P_{S'}(I) \subseteq P_S(I)$$

for every $X \subseteq U$.

8. Uncertainty measure

In order to describe to what extent the probability of an event $X \subseteq U$ can be evaluated in the given stochastic approximation space $S$ we introduce uncertainty measure $\zeta_S(X)$ defined as below:

$$\zeta_S(X) = \overline{P}_S(X) - P_S(X),$$

which is simply the length of the interval $P_{S'}(I)$.

By simple calculation one can show the following

(D1) $\zeta_{S\{X\}} = \zeta_S(X)$

(D2) $\zeta_S(I \cup J) \leq \zeta_S(I) + \zeta_S(J)$

(D3) $\zeta_S(I \cap J) \geq \zeta_S(I) \cdot \zeta_S(J)$

(D4) If $S'$ is finer than $S$, then

$$\zeta_{S'}(I) \leq \zeta_S(I).$$

9. Example

Consider a stochastic approximation space with the set of primitive events $U = \{X_0, \ldots, X_{10}\}$ and the indiscernibility relation defined by the following equivalence classes:

$$E_1 = \langle X_0, x_1 \rangle$$
$$E_2 = \langle x_2, x_6, x_9 \rangle$$
$$E_3 = \langle x_3, x_5 \rangle$$
$$E_4 = \langle x_4, x_8 \rangle$$
$$E_5 = \langle x_7, x_{10} \rangle$$

We assume that we are given the probabilities of the atomic events:

$$P(E_1) = 1/4$$
$$P(E_2) = 1/8$$
$$P(E_3) = 3/8$$
$$P(E_4) = 1/8$$
$$P(E_5) = 1/8.$$

For observable events in the space

$$X_1 = \langle X_0, X_1, X_4, X_8 \rangle = E_1 \cup E_4$$
$$Y_1 = \langle X_2, X_5, X_7, X_9 \rangle = E_2 \cup E_5$$
$$Z_1 = \langle X_1, X_3, X_5, X_6, X_9 \rangle = E_2 \cup E_3$$

We have the following probabilities

$$P(X_1) = 1/4 + 1/8 = 3/8$$
$$P(Y_1) = 3/8 + 1/8 = 1/2$$
$$P(Z_1) = 1/8 + 3/8 = 1/2$$

For roughly observable events in the space
\[ X_2 = \langle x_0, x_1, x_2, x_3, x_4, x_7 \rangle \]
\[ Y_2 = \langle x_1, x_2, x_3, x_4, x_9, x_10 \rangle \]
\[ Z_2 = \langle x_2, x_3, x_4, x_8 \rangle \]

we have the following lower and upper approximations

\[ \overline{A}(X_2) = E_3 \cup E_4 \]
\[ \underline{A}(X_2) = E_1 \cup E_3 \cup E_4 \cup E_5 \]
\[ \overline{A}(Y_2) = E_3 \]
\[ \underline{A}(Y_2) = E_1 \cup E_4 \cup E_5 \]
\[ \overline{A}(Z_2) = E_2 \cup E_3 \cup E_4 \]
\[ \underline{A}(Z_2) = E_2 \cup E_3 \cup E_4 \]

Inner and outer probabilities of these sets are the following:

\[ P(X_2) = 3/8 + 1/8 = 1/2 \]
\[ P(X_2) = 1/4 + 3/8 + 1/8 + 1/8 = 7/8 \]
\[ P(Y_2) = 1/8 \]
\[ P(Y_2) = 1/4 + 1/8 + 1/8 = 1/2 \]
\[ P(Z_2) = 1/8 \]
\[ P(Z_2) = 1/8 + 3/8 + 1/8 = 5/8 \]

and consequently the rough probabilities of these sets are

\[ P^*(X_2) = <1/2, 7/8> \]
\[ P^*(Y_2) = <1/8, 1/2> \]
\[ P^*(Z_2) = <7/8, 5/8> \]

For internally unobservable events in the space

\[ X_3 = \langle x_0, x_1, x_2, x_3, x_4, x_7 \rangle \]
\[ Y_3 = \langle x_1, x_2, x_3, x_4, x_9, x_10 \rangle \]
\[ Z_3 = \langle x_0, x_2, x_3, x_4, x_6, x_10 \rangle \]

we have the following lower approximations:

\[ \overline{A}(X_3) = E_1 \]
\[ \underline{A}(Y_3) = E_2 \]
\[ \underline{A}(Z_3) = E_4 \]

and the corresponding rough probabilities are

\[ P^*(X_3) = <1/4, 1> \]
\[ P^*(Y_3) = <1/8, 1> \]
\[ P^*(Z_3) = <1/8, 1> \]

For internally unobservable events in the space

\[ X_4 = \langle x_0, x_2, x_3 \rangle \]
\[ Y_4 = \langle x_1, x_2, x_4, x_7 \rangle \]
\[ Z_4 = \langle x_2, x_3, x_4 \rangle \]

we have the following upper approximations:

\[ \overline{A}(X_4) = E_1 \cup E_2 \cup E_3 \]
\[ \overline{A}(Y_4) = E_1 \cup E_2 \cup E_4 \cup E_5 \]
\[ \overline{A}(Z_4) = E_2 \cup E_3 \cup E_4 \]

which gives the following outer probabilities

\[ \overline{T}(X_4) = 1/4 + 1/8 + 3/8 = 3/4 \]
\[ \overline{T}(Y_4) = 1/4 + 1/8 + 1/8 + 1/8 = 5/8 \]
\[ \overline{T}(Z_4) = 1/8 + 3/8 + 1/8 = 5/8 \]

For externally unobservable events in the space
and consequently
\[ P^*(X_4) = <0, 3/4 > \]
\[ P^*(Y_4) = <0, 5/8 > \]
\[ P^*(Z_4) = <0, 5/8 > \]

For totally unobservable sets
\[ X_5 = \langle x_0, x_2, x_3, x_4, x_7 \rangle \]
\[ Y_5 = \langle x_1, x_5, x_6, x_8, x_{10} \rangle \]
\[ Z_5 = \langle x_0, x_2, x_5, x_7, x_8 \rangle \]

we get
\[ P^*(X_5) = <0, 1 > \]
\[ P^*(Y_5) = <0, 1 > \]
\[ P^*(Z_5) = <0, 1 > \]

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References