Rough Sets and Multiexpert Systems

by

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Summary. In this paper the expert systems are considered as a special case of information systems and it is shown that the rough sets approach to multiexpert system analysis yields new and fruitful results.

1. Introduction. In this paper multiexpert systems are considered. We assume that the knowledge of every expert is identified with the expert's ability to classify objects of a certain universe. We show that this class of expert systems can be regarded as a special case of information systems (see [1]) and can easily be analyzed in terms of rough set concepts (see [2, 3]).

2. Multiexpert systems. A multiexpert system is a 4-tuple

\[ S = (U, A, V, f) \]

where

- \( U \) is a set called the universe
- \( A = D \cup E \) \((D \cap E = \emptyset)\) is the set of attributes; \( D \) is the set of description attributes and \( E \) - experts attributes
- \( V = \bigcup_{a \in A} V_a \) is the domain (set of values) of attribute \( a \)
- \( f : U \times A \rightarrow V \) is a total function, such that \( f(x, a) \in V_a \), for every \( x \in U \) and \( a \in A \)

In what follows, we shall identify experts' attributes with experts, i.e., if \( a \in E \) then \( a \) will be called an expert.
The function \( f_a : U \to V_a \) such that \( f_a(x) = f(x, a) \) for every \( a \in E \) is called the opinion of an expert \( a \).

If \( f(x, a) = f(y, a) \) for every \( a \in B, B \subseteq A \) we say that \( x \) and \( y \) are indiscernible with respect to \( B \) in \( S \). Thus, each \( B \subseteq A \) generates an indiscernibility relation \( \tilde{B} \), such that \((x, y) \in \tilde{B}\) if \( x \) and \( y \) are indiscernible with respect to \( B \) in \( S \). Obviously, \( \tilde{B} \) is an equivalence relation, for every \( B \). The family of equivalence classes of \( \tilde{B} \) is denoted by \( B^* \) and equivalence classes of \( \tilde{B} \) are called \( B \)-elementary sets in \( S \).

We say that set \( B \subseteq A \) is independent in \( S \), if for every \( C \subseteq B, C \sim \tilde{B} \), otherwise \( B \) is dependent in \( S \).

The maximal independent subset of \( B \) is called reduct of \( B \) in \( S \). A reduct of \( B \) is equal to \( B \) we say that \( B \) is reduced and the corresponding system is said to be \( B \)-reduced. Thus, we can reduce both set \( D \) and set \( E \). Reducing the set \( D \) we simplify the system with regard to the descriptions of objects, and the reduction of set of experts \( E \) provides the minimal set of experts with the same decisive power as the whole set of experts \( E \).

The classification \( a^*, a \in E \) is called the expert's decision and the classification \( B^*, B \subseteq E \) is called the decision of group of experts \( B \).

3. Approximations. Let \( S = (U, A, V, f) \) be an multiexpert system, \( X \subseteq U \) and \( B \subseteq A \).

We define two sets

\[
BX = \{ x \in U : [x]_B \subseteq X \}
\]

\[
\tilde{BX} = \{ x \in U : [x]_B \cap X \neq \emptyset \}
\]

called the \( B \)-lower and \( B \)-upper approximation of \( X \) in \( S \).

The set \( B_{\tilde{B}}(X) = \tilde{BX} - BX \) is called the \( B \)-boundary of \( X \) in \( S \).

The number

\[
\mu_B(X) = \frac{\text{card } BX}{\text{card } B\tilde{X}}
\]

is called the accuracy of approximation of \( X \) by \( B \) in \( S \).

Let \( F = \{ X_1, X_2, \ldots, X_m \} \) be a family of subsets of \( U \), i.e. \( X_i \subseteq U \) for every \( i, 1 \leq i \leq m \). By \( B \)-lower and \( B \)-upper approximation of \( F \) we mean the sets

\[
BF = \{ BX_1, BX_2, \ldots, BX_m \}
\]

\[
\tilde{BF} = \{ \tilde{BX}_1, \tilde{BX}_2, \ldots, \tilde{BX}_m \}
\]

The number
\[ \beta_B(F) = \frac{\text{card } \bigcup_{i=1}^{n} BX_i}{\sum_{i=1}^{n} \text{card } \bar{B}X_i} \]

is called the accuracy of the approximation of \( F \) by \( B \) in \( S \) and the number

\[ \gamma_n(F) = \frac{\text{card } \bigcup_{i=1}^{n} BX_i}{\text{card } U} \]

is called the quality of the approximation of \( F \) by \( B \) in \( S \).

4. Dependency of attributes. Let \( S = (U, A, V, f) \) be a multiexpert system, \( k \) — a number such that \( 0 \leq k \leq 1 \), and \( B, C \subseteq A \).

We say that \( C \) depends in a degree \( k \) on \( B \) in \( S \), in symbols \( B \downarrow_k C \), if \( k = \gamma_B(C^*) \). If \( k = 1 \) we write \( B \rightarrow C \) instead of \( B \downarrow C \), and we say that \( C \) totally depends on \( B \) in \( S \); if \( 0 < k < 1 \) we say that \( C \) roughly depends on \( B \) in \( S \) and if \( k = 0 \) we say that \( C \) is totally independent on \( B \) in \( S \).

If \( D \rightarrow E \) we say that the multiexpert system is deterministic; otherwise the multiexpert system is nondeterministic.

It means that if the multiexpert system is deterministic we are able to describe experts decisions exactly by the description attributes available in the multiexpert system; if the multiexpert system is nondeterministic we are able to describe experts decisions only with a certain approximation (roughly), using the description attributes available in the system.

The numbers \( \beta_D(E^*) \) and \( \gamma_D(E^*) \) are called the accuracy and the quality of the multiexpert system, respectively. In other words, these numbers say how precise the experts decisions can be described in terms of attributes available in the system.

If the set of experts \( E \) is dependent in a multiexpert system \( S \), and \( C \) is a reduct of \( E \) in \( S \) then \( C \rightarrow E-C \), which is to mean that the set of experts \( E-C \) is superfluous in \( S \).

Property 1. If \( B \subseteq D \) and \( C \subseteq E \) are reducts of \( D \) and \( E \), respectively, then

\[ \gamma_D(E^*) = \gamma_B(C^*) \]
\[ \beta_D(E^*) = \beta_B(E^*) \]

Let \( B \subseteq D \) and \( C \subseteq E \). The set

\[ \text{Pos}_B(C^*) = \bigcup_{i=1}^{n} BX_i \]
where \( C^* = \{X_1, X_2, \ldots, X_n\} \) will be called the positive region of the classification \( C^* \) with respect to \( B \) (B-positive region) in \( S \).

The set

\[
B_{\text{pos}}(C^*) = \bigcup_{i=1}^{n} B X_i
\]

is called the doubtful region of the classification \( C^* \) with respect to \( B \) (B-doubtful region) in \( S \).

Positive region of the classification is the subset of the universe on which the experts decisions agree, and can be described in terms of attributes considered; the doubtful region of the classification is the subset of the universe on which it is impossible to recognize as to the experts opinion, whether they agree or disagree – employing the attributes of the system.

Of course,

\[
\text{Pos}_B(C^*) \cup B_{\text{pos}}(C^*) = U
\]

for every \( B \subseteq D \) and \( C \subseteq E \).

5. Rough equality of experts. In this section we shall compare sets of experts using the concepts introduced previously.

Let \( S = (U, A, V, f) \) be a multiexpert system, \( A = D \cup E \), \( B \subseteq D \) and \( E_1, E_2 \subseteq E \).

We shall employ the following definitions

A1) \( E_1 \) and \( E_2 \) are bottom equal with respect to \( B \) in \( S \), in symbols

\( E_1 \simeq_B E_2 \) if \( B(E_1^+) = B(E_2^+) \)

A2) \( E_1 \) and \( E_2 \) are top equal with respect to \( B \) in \( S \), in symbols

\( E_1 \equiv_B E_2 \) if \( B(E_1^*) = B(E_2^*) \)

A3) \( E_1 \) and \( E_2 \) are roughly equal with respect to \( B \) in \( S \), in symbols

\( E_1 \approx_B E_2 \) if \( E_1 \simeq_B E_2 \) and \( E_1 \equiv_B E_2 \).

The bottom equality preserves the positive region of the classifications \( E_1^+ \) and \( E_2^+ \); the top equality preserves the upper approximation of each class in both classifications \( E_1^+ \) and \( E_2^+ \), and the rough equality preserves the doubtful region of the classification \( E_1^* \) and \( E_2^* \).

Property 2.

B1) If \( E_1 \approx_B E_2 \) then \( \gamma_B(E_1^+) = \gamma_B(E_2^+) \)

B2) If \( E_1 \equiv_B E_2 \) then \( \beta_B(E_1^*) = \beta_B(E_2^*) \)

B3) If \( E_1 \approx_B E_2 \) then \( \gamma_B(E_1^+) = \gamma_B(E_2^+) \) and \( \beta_B(E_1^*) = \beta_B(E_2^*) \).

6. Rough inclusion of experts. In this section we shall compare sets of experts with respect to their decisive “power”. In order to do so, we shall employ the following definitions.
Let $S = (U, A, V, f)$ be a multieexpert system, $A = D \cup E$, $B \subseteq D$ and let $F = \{X_1, X_2, \ldots, X_m\}$, $G = \{Y_1, Y_2, \ldots, Y_m\}$, $X_i, Y_i \subseteq U$ be two families of subsets of $U$.

We shall use the following notations:
C1) $\sqsubseteq_B G$ if $BX_i \subseteq BY_i$
C2) $F \sqsubseteq_B G$ if $BX_i \subseteq BY_i$
C3) $F \sqsubseteq_B G$ if $F \sqsubseteq_B G$ and $F \sqsubseteq_B G$

for every $i$, $1 \leq i \leq m$.

It is easy to see that all three relations defined above are the partial ordering relations.

Let $E_1, E_2 \subseteq E$. We say that
D1) The set of experts $E_1$ is **internally better** then the set of experts $E_2$ with respect to $B$ in $S$ if $E_1 \sqsubseteq_B E_2$.
D2) The set of experts $E_1$ is **externally better** then the set of experts $E_2$ with respect to $B$ in $S$ if $E_1 \sqsubseteq_B E_2$.
D3) Set of experts $E_1$ is **roughly better** than the set of experts $E_2$ with respect to $B$ in $S$ if $E_1 \sqsubseteq_B E_2$.

There is also possible another comparison of experts

E1) Set of experts $E_1$ is **internally finer** than the set of experts $E_2$ with respect to $B$ in $S$ in symbols $E_1 \simeq_B E_2$, if $\gamma_B(E_1) > \gamma_B(E_2)$.
E2) Set of experts $E_1$ is **externally finer** than the set of experts $E_2$ with respect to $B$ in $S$ in symbols $E_1 \simeq_B E_2$ if $\beta_B(E_1) < \beta_B(E_2)$.
E3) Set of experts $E_1$ is **roughly finer** than the set of experts $E_2$ with respect to $B$ in $S$ in symbols $E_1 \simeq_B E_2$, if $E_1 \simeq_B E_2$ and $E_1 \simeq_B E_2$.

**Property 3.**

a) If $E_1 \sqsubseteq_B E_2$ then $E_1 \simeq_B E_2$.
b) If $E_1 \sqsubseteq_B E_2$ then $E_1 \simeq_B E_2$.
c) If $E_1 \sqsubseteq_B E_2$ then $E_1 \simeq_B E_2$.

The converse implication is not true.

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REFERENCES